# Univalent Functions and Some Related Concepts[1]

Shahpour Nosrati
*shahpournosrati@gmail.com*

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Most of the results in this context are no more than ten years old and many parts of the theory are still developing and have not yet found a final form. Although it was impossible to include every result in this field, so, I have tried to give a fairly complete survey of the available material.

## 0.1 Introduction

In this context, $\mathbb{C}$ is complex plane, $\hat{\mathbb{C}}$ is extended complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is open unit disk and it's boundary denotes by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Every connected open set named domain will denote here by $D$ or $\Omega$. $\mathbf{H(\Omega)}$ denotes the set of all analytic (Holomorphic) functions defined in a domain $\Omega$. The function

$$\begin{cases} f : D \subset \mathbb{C} \to \mathbb{C} \\ f(x, y) = u(x, y) + iv(x, y) \end{cases}$$

is analytic on $D$ when $f$ is continuous and $u$ and $v$ satisfy Cauchy-Riemann equations, those are $u_x = v_y$ and $u_y = -v_x$. This is equivalent to $f \in C^1(D)$ if the real functions $u = \mathbf{Re}\, f$ and $v = \mathbf{Im} f$ of the real variables $x$ and $y$ have continuous first order partial derivatives in $D$.

Every analytic map is *conformal* if it's derivative never vanishes. A conformal map preserves angle and direction when it maps intersection curves in a domain to intersection curves in it's range.

**Theorem 0.1.1.** *(Riemann Mapping Theorem [62]) Let $G$ be a simply-connected, proper subset of the complex plane with $z_0 \in G$. Then there exists a unique univalent, onto analytic function $\phi : G \to \mathbb{D}$ such that $\phi(z_0) = 0$ and $\phi'(z_0) > 0$.*

Let $G \neq \mathbb{C}$ be a simply-connected domain, we may replace the function $F : G \to \mathbb{C}$ by the function $f := Fo\phi^{-1} : \mathbb{D} \to \mathbb{C}$, where $\phi : G \to \mathbb{D}$ is Riemann map. Therefore, in the study of univalent analytic functions, we can restrict our attention to functions defined on $\mathbb{D}$.

# Chapter 1

# Univalent analytic functions

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $\mathbb{D}$. Let $\hat{\mathcal{A}}$ be the subclass of $\mathcal{A}$ consisting of functions $f$ normalized by $f(0) = 0$, $f'(0) = 1$. Further, let $\mathcal{S}$ denotes the class of functions $f \in \hat{\mathcal{A}}$ that are univalent[1] in $\mathbb{D}$. The class $\mathcal{S}$ is a compact family. This class is preserved under conjugation, rotation, dilation, disc automorphism[2] and range transformation [24]. An example of a function of class $\mathcal{S}$ is the Koebe function

$$k_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + \cdots \tag{1.2}$$

This function maps the disk $\mathbb{D}$ onto the entire plane minus the part of the negative real axis from $-\frac{1}{4}$ to infinity (Fig.1.1). It's extremal function for class $\mathcal{S}$ also. The familiar Koebe one-quarter theorem [24] says:

[1] called *schlicht,* injective, *one-to-one also.*

[2] For a given locally univalent analytic function $f \in \mathcal{A}$, the disk automorphism is the function $\Lambda_f : \mathbb{D} \to \mathbb{C}$ given by

$$\Lambda_f(z) = \frac{f(\frac{z+z_0}{1+z\overline{z_0}}) - f(z_0)}{(1-|z_0|^2) f'(z_0)}$$

where $z_0 \in \mathbb{D}$. A family $\mathcal{F}$ is linearly invariant if for every $f \in \mathcal{F}$ we have $\Lambda_f(z) \in \mathcal{F}$.

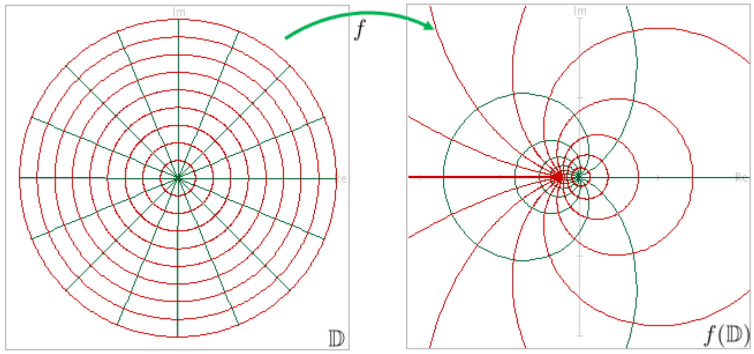


Figure 1.1: The image of $\mathbb{D}$ under Koebe map.

**Theorem 1.0.1.** *The image of $\mathbb{D}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.*

In earlier of 20th century on univalent functions discution, Bieberbach conjectured that for every function $f \in \mathcal{S}$, coefficient may satisfy the condition $|a_n| \leq n$ for all $n \geq 2$. The coefficients bound obtained as following [36]:

| Name | Year | $\lvert a_n\rvert < Cn$ | |
|---|---|---|---|
| Littlewood | 1923 | $\lvert a_n\rvert < en$ | $\simeq$ 2.7183n |
| Landau | 1929 | $\lvert a_n\rvert < \left(\frac{1}{2}+\frac{1}{\pi}\right) en$ | $\simeq$ 2.2244n |
| Goluzin | 1946 | $\lvert a_n\rvert < \frac{3}{4} en$ | $\simeq$ 2.0388n |
| Bazilevič | 1947 | $\lvert a_n\rvert < \frac{9}{4}\left(\frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}dx + 0.2649\right) n$ | $\simeq$ 1.9240n |
| Milin | 1949 | $\lvert a_n\rvert < \frac{1}{2} en + 1.80$ | $\simeq$ 1.3592n+ 1.80 |
| Bazilevič | 1949 | $\lvert a_n\rvert < \frac{1}{2} en + 1.51$ | $\simeq$ 1.3592n+ 1.51 |
| Milin | 1964 | $\lvert a_n\rvert < \frac{\sqrt{e^{1.6}-1}}{1.6} n$ | $\simeq$ 1.2427n |
| FitzGerald | 1971 | $\lvert a_n\rvert < \sqrt{\frac{7}{6}} n$ | $\simeq$ 1.0802n |
| Horowitz | 1975 | $\lvert a_n\rvert < \sqrt[6]{\frac{209}{140}} n$ | $\simeq$ 1.0691n |

also some primary coefficients bound obtained as following

| $\lvert a_n\rvert \leq n$ | Year | Name |
|---|---|---|
| $\lvert a_2\rvert \leq 2$ | 1916 | Bieberbach [11]. |
| $\lvert a_3\rvert \leq 3$ | 1923 | Lowner [46]. |
| $\lvert a_4\rvert \leq 4$ | 1955 | Garabedian and Schiffer [29]. |
| $\lvert a_5\rvert \leq 5$ | 1972 | Pederson and Schiffer [59]. |
| $\lvert a_6\rvert \leq 6$ | 1968 | Pederson [58] and independently Ozawa [57] proved it in 1972. |

The best result known was due to D. Horowitz [36] who proved that $|a_n| < 1.0657n$ using a very deep method due to Carl FitzGerald [27]. Finally, the conjecture of Bieberbach proved by Louis de Branges in 1985:

**Theorem 1.0.2.** *(de Branges Theorem [19]) For every $f \in \mathcal{S}$, coeffiecients satisfy $|a_n| \leq n$, for all $n$.*

de Branges proved the Milin conjecture (1971) on logarithmic coefficients, which implies the Robertson conjecture (1936) on odd univalent functions, and it implies the Rogosinski conjecture (1943) on subordinate functions, and finally the Bieberbach conjecture ([24], p.197). Milin's conjecture asserts that the logarithmic coefficients $\gamma_n$ of a univalent function of the form (1.1) defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty}\gamma_n z^n \tag{1.3}$$

satisfy the inequality

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0\ ,\ \ n = 1, 2, \cdots \tag{1.4}$$

Clearly, the logarithmic coefficients of the Koebe function are $\gamma_n = \dfrac{1}{n}$ and satisfy the Milin's conjecture. The Robertson's conjecture says that for each odd univalent function like $h(z) = z + c_3 z^3 + c_5 z^5 + \cdots$ the inequality

$$1 + |c_3|^2 + \cdots + |c_{2n-1}|^2 \leq n$$

holds [7]. Rogosinski conjecture will state in lemma 1.1.1.

## 1.1 Subordination

We say that $f$ is subordinate to $g$ and write $f \prec g$ or $f(z) \prec g(z)$ for it if there exists an analytic function $\omega$ on $\mathbb{D}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$

and $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. When $g$ is univalent, $f$ is subordinate to $g$ precisely if $f(0) = g(0)$ and if $f(\mathbb{D}) \subset g(\mathbb{D})$.

**Lemma 1.1.1.** *(Rogosinski Conjecture [24]) If $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is analytic in $\mathbb{D}$ and $g \prec f$ for some $f \in \mathcal{S}$, then $|b_n| \leq n$ for $n = 1, 2, \cdots$.*

This has known as the generalized Bieberbach conjecture. If $f, g \in \mathcal{S}$ and $g \prec f$ then $f = g$.

## 1.2 Starlikeness

An analytic function $f(z)$ is said to be starlike if it's range is starlike with respect to the origin. In geometric view of the range of $f(z)$, this means that every point of the range can be connected to the origin by a radial line that lies entirely in the region. In the other words, $\arg\{f(e^{i\theta})\}$ will be a nondecreasing function of $\theta$, or that

$$\frac{\partial}{\partial\theta} \arg\{f(e^{i\theta})\} \geq 0$$

Starlikeness is a hereditary property for conformal mappings, so if $f$ is analytic and univalent in $\mathbb{D}$ with $f(0) = 0$, and if $f$ maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin, then the image of every subdisk $|z| < r < 1$ is also starlike with respect to the origin. The class of all starlike functions in $\mathcal{S}$ is shown by $\mathcal{S}^*$. Furthermore $f(z) \in \mathcal{S}^*$ iff $\mathbf{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0$, iff $z\frac{f'(z)}{f(z)} \prec \frac{1+z}{1-z}$. Such function with $\mathbf{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0$ is univalent on $\mathbb{D}$.

Clearly, the multiply of two starlike functions is starlike, because if $G(z) = f(z)g(z)$ then $\log G(z) = \log f(z) + \log g(z)$, so with differentiation we have $z\frac{G'(z)}{G(z)} = z\frac{f'(z)}{f(z)} + z\frac{g'(z)}{g(z)}$ for $z \in \mathbb{D}$, and it's done.

The extremal function for class $\mathcal{S}^*$ is $k_0(z) = \frac{z}{(1-z)^2}$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be starlike on $\mathbb{D}$, then there is a uniquely determined probability measure $\mu$ defined on the Borel subsets of the unit circle

$\mathbb{T}$ such that

$$F(z) = z\frac{f'(z)}{f(z)} = \int \frac{1+\overline{\gamma}z}{1-\overline{\gamma}z}d\mu(\gamma) \quad , \quad z \in \mathbb{D}$$

then

$$F(z) = 1 + 2\sum_{n=1}^{\infty} z^n \int \overline{\gamma}^n d\mu(\gamma)$$

Defining the sequence $c_n = 2\int \overline{\gamma}^n d\mu(\gamma)$ it follows quickly that

$$F(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$f(z) = z\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} c_n z^n\right)$$

and that the sequences $a_n$ and $c_n$ are linked by means of the following of the form

$$(n+1)a_n = \sum_{k=1}^{n} a_k c_{n-k}$$

for $n = 1, 2, 3, \cdots$, where $a_1 = 1$.

**Lemma 1.2.1.** *(Bazilevič [10]) If $f(z)$ is starlike and $\gamma > 0$ then*

$$\left(\frac{f(z)}{z}\right)^{\gamma} \prec \frac{|f'(0)|^{\gamma}}{(1-z)^{2\gamma}}$$

Robertson (1936) introduced a subclass of $\mathcal{S}$,

**Definition 1.2.1.** *(Robertson [68]) An analytic function $f(z)$ is called starlike of order $\alpha$ with $0 \leq \alpha < 1$ satisfying the inequality*

$$\mathbf{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \alpha$$

*for $z \in \mathbb{D}$ and the set of all such functions is denoted by $\mathcal{S}^*(\alpha)$.*

In this definition $\alpha$ is restricted with $0 \leq \alpha < 1$ because $\mathbf{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \alpha$ with $\alpha < 0$ may fail to be univalent on $\mathbb{D}$. Markes, Robertson and Scott (1961) obtained starlike products that functions which are starlike of positive order and his results follows:

**Lemma 1.2.2.** *[47] Let $f_n(z)$ for $n = 1, 2, \cdots, N$, be starlike at least of order $1 - d_n \geq 0$, where $d_n \geq 0$, and let $s_N = 1 - \sum_{n=1}^{N} d_n \geq 0$. Then the product*

$$F_N(z) = z \prod_{n=1}^{N} \frac{f_n(z)}{z} \tag{1.5}$$

*is starlike at least of order $s_N$. There exist functions of this type which are not starlike of any order greater than $s_N$.*

**Lemma 1.2.3.** *[47] Let $f(z) \in \mathcal{S}$ and $F(z) = z\Big(\dfrac{f(z)}{z}\Big)^{\alpha}$ with $F'(0) = 1$, then for $\alpha \geq 1$, $F(z)$ is univalent and starlike in $\mathbb{D}$ if and only if $f(z)$ is starlike at least of order $1 - \dfrac{1}{\alpha}$.*

This lemma for $0 \leq \alpha < 1$ leads to $f(z) = z\Big(\dfrac{F(z)}{z}\Big)^{\frac{1}{\alpha}}$.

**Lemma 1.2.4.** *[47, 80] The function $f \in \mathcal{A}$, is starlike at least of order $\alpha$, where $0 \leq \alpha < 1$, if*

$$\sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} |a_n| \leq 1 \tag{1.6}$$

*If $a_n \leq 0$ for all $n$, then (1.6) is a necessary condition for $f(z)$ to be starlike at least of order $\alpha$.*

**Example 1.2.1.** *[47] The function $f_n(z) = z - \dfrac{z^3}{4n^2}$ is by Lemma 1.2.4 starlike at least of order $a_n = 1 - d_n = 1 - \dfrac{2}{4n^2 - 1}$. Since $s_N = 0$ Lemma 1.2.2 shows that*

$$\frac{2}{\pi} \sin \frac{\pi}{2} z = z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{4n^2}\Big) \tag{1.7}$$

*is univalent and starlike in $\mathbb{D}$.*

**Example 1.2.2.** *[47] The function $\dfrac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \dfrac{z}{n}\Big) e^{-\frac{z}{n}}$ where $\Gamma(z)$ is the Euler gamma function, is univalent and starlike for $|z| < r_0$, where*

*$r_0$ is the modulus of the largest negative zero $= -0.50\cdots$ of $\Gamma'(z)$, and the result is sharp.*

The radius of starlikeness is less than $\tanh\dfrac{\pi}{4} \approx 0.655\cdots$. Another subclass of starlike class was introduced by Stankiewicz (1966) and Brannan (1969):

**Definition 1.2.2.** *[84, 12] An analytic function $f(z)$ is called strongly starlike of order $\beta$ with $(0 < \beta \leq 1)$ satisfying the inequality*

$$\left|\arg\left(z\frac{f'(z)}{f(z)}\right)\right| < \beta\frac{\pi}{2}$$

*for $z \in \mathbb{D}$ and the set of all such functions is denoted by $\mathcal{S}^*[\beta]$.*

## 1.3 Convexity

An analytic function $f(z)$ is said to be convex if it's range is a convex set. In geometric view of the range of $f(\mathbb{D})$, this means that $\arg\left\{\dfrac{\partial}{\partial\theta}f(e^{i\theta})\right\}$ be a nondecreasing function of $\theta$, or that

$$\frac{\partial}{\partial\theta}\arg\left\{\frac{\partial}{\partial\theta}f(e^{i\theta})\right\} \geq 0$$

Convexity is a hereditary property for conformal mappings. If $f$ is analytic and univalent in $\mathbb{D}$ and maps it onto a convex domain, then the image of every subdisk $|z| = r < 1$ is also convex. So we may say for every $r < 1$

$$\frac{\partial}{\partial\theta}\arg\{\frac{\partial}{\partial\theta}f(re^{i\theta})\} \geq 0$$

for $0 \leq \theta \leq 2\pi$. The radius of convexity for the class $\mathcal{S}$ is $2-\sqrt{3} \approx 0.267\cdots$. The class of all convex functions in $\mathcal{S}$ is shown by $\mathcal{K}$. This class is linearly invariant, furthermore $f(z) \in \mathcal{K}$ iff $\mathbf{Re}\left\{1 + z\dfrac{f''(z)}{f'(z)}\right\} > 0$, iff $1 + z\dfrac{f''(z)}{f'(z)} \prec \dfrac{1+z}{1-z}$. It is clear that every convex function is starlike. The extremal function for class $\mathcal{K}$ is $\ell(z) = \dfrac{z}{1-z}(Fig.1.2)$.

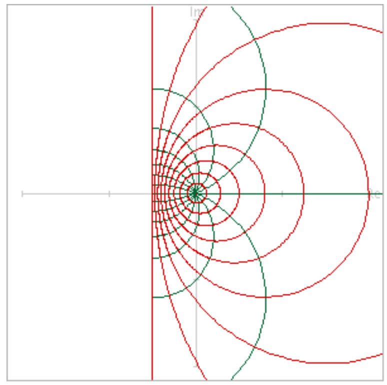

Figure 1.2: The image of $\mathbb{D}$ under convex map $f = \frac{z}{1-z}$.

**Lemma 1.3.1.** *(Silverman [80]) Let $f(z)$ be alalytic in $\mathbb{D}$ of the form (1.1). If $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$, then $f \in \mathcal{K}$.*

A theorem by Alexander (1915) shows a relation between classes $\mathcal{S}^*$ and $\mathcal{K}$:

**Theorem 1.3.1. (Alexander's Theorem)** *Let $f$ be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{K}$ iff $zf'(z) \in \mathcal{S}^*$.*

so, if $f \in \mathcal{S}^*$ then $g(z) = \int_0^z \frac{f(z)}{z} dz$ is convex. A subfamily of $\mathcal{K}$, denoted by $\mathcal{K}(\alpha)$, consisting of convex functions of order $\alpha$ introduced by Robertson [68]. Here, for a constant $0 \leq \alpha < 1$ function f in $\hat{\mathcal{A}}$ is called *convex of order* $\alpha$ if

$$\mathbf{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > \alpha \tag{1.8}$$

for $z \in \mathbb{D}$. It is obvious that for $0 \leq \alpha < \beta < 1$,

$$\mathcal{K}(\beta) \subset \mathcal{K}(\alpha) \subset \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \tag{1.9}$$

The extremal function for class $\mathcal{K}(\alpha)$ is

$$k_\alpha(z) = \begin{cases} \frac{(1-z)^{2\alpha-1}-1}{1-2\alpha} & ; \quad \alpha \neq \frac{1}{2}, \\ -\log(1-z) & ; \quad \alpha = \frac{1}{2}. \end{cases} \tag{1.10}$$

which maps $\mathbb{D}$ univalently onto the half-plane $\mathbf{Re}\{w\} > \alpha$ (Sugawa [86]). It's interested to know that $1 + z\dfrac{k''_\alpha(z)}{k'_\alpha(z)} = \dfrac{1+(1-2\alpha)z}{1-z}$ with $k_\alpha(0) = 0$, $k'_\alpha(0) = 1$. Clearly $k_0(z) = \dfrac{z}{1-z}$ which maps $\mathbb{D}$ univalently onto the half-plane $\mathbf{Re}\{w\} > -\frac{1}{2}$, and $\dfrac{k_0(z)}{z} = \dfrac{1}{1-z}$ maps $\mathbb{D}$ univalently onto the half-plane $\mathbf{Re}\{w\} > \frac{1}{2}$. By Alexander theorem (1.3.1) we see:

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \tag{1.11}$$

Furthermore,

**Lemma 1.3.2.** *(Sim and Kwon [81]) If $f(z) \in \mathcal{K}(\alpha)$ then $\mathbf{Re}\sqrt{f'(z)} > \dfrac{1}{2-\alpha}$.*

and more,

**Lemma 1.3.3.** *(Sim and Kwon [81]) If $\mathbf{Re}\left\{1 + z\dfrac{f''(z)}{f'(z)}\right\} < \beta$ with $1 < \beta < 2$ then $\mathbf{Re}\sqrt{f'(z)} < \dfrac{1}{2-\beta}$.*

Is there any relation between classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$?. At the first we know that a normalized convex function in the unit disk is known to be starlike at least of order $\dfrac{1}{2}$ [48] and then, a corollary by MacGregor says:

**Lemma 1.3.4.** *[89] Let $0 \le \alpha < 1$ and $f \in \mathcal{K}(\alpha)$, then $f(z) \in \mathcal{S}^*(\delta(\alpha))$, where*

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & ; \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & ; \quad \alpha = \frac{1}{2}. \end{cases} \tag{1.12}$$

**Lemma 1.3.5.** *[80] The function $f \in \mathcal{A}$, is convex of order $\alpha$, where $0 \le \alpha < 1$, if*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| \le 1 \tag{1.13}$$

**Lemma 1.3.6.** *(Marx and Strohhacker 1933) For a normalized convex function $f$ on the unit disk $\mathbb{D}$, $\dfrac{f(z)}{z} \prec \dfrac{1}{1-z}$ for $z \in \mathbb{D}$.*

**Lemma 1.3.7.** *(Brickman, Hallenbeck, MacGregor and Wilken 1973) Let $\frac{1}{2} \leq \alpha < 1$. Then, for $f \in \mathcal{K}(\alpha)$, we have $\dfrac{f(z)}{z} \prec \dfrac{k_\alpha(z)}{z}$ for $z \in \mathbb{D}$, Furthermore $\dfrac{k_\alpha(-r)}{-r} \leq \mathbf{Re}\,\dfrac{f(z)}{z} \leq \dfrac{k_\alpha(r)}{r}$ for $|z| = r < 1$.*

**Lemma 1.3.8.** *(Sugawa and Wang 2015 [86]) Let $0 < \alpha < 1$. Then, for $f \in \mathcal{K}(\alpha)$, we have $\dfrac{f(z)}{z} \prec \dfrac{k_\alpha(z)}{z}$ for $z \in \mathbb{D}$.*

**Lemma 1.3.9.** *(Styer and Wright 1973) Let $f, g \in \mathcal{K}$ and $|\mathbf{Im}\dfrac{f(z)}{z}| \leq \dfrac{\pi}{4}$ and $|\mathbf{Im}\dfrac{g(z)}{z}| \leq \dfrac{\pi}{4}$ on $\mathbb{D}$ then $\dfrac{f+g}{2} \in \mathcal{S}^*$.*

**Lemma 1.3.10.** *(Hallenbeck and Ruscheweyh 1975 [34]) for $f, g \in \mathcal{K}$ and $f''(0) = g''(0) = 0$ we have $\dfrac{f+g}{2} \in \mathcal{S}^*$.*

They also proved that for $f \in \mathcal{K}$ with $f''(0) = 0$ which satisfies $|\mathbf{Im}\dfrac{f(z)}{z}| \leq \dfrac{\pi}{4}$ we have

$$\frac{f(z)}{z} \prec H_1(z) := \frac{1}{2\sqrt{z}} \log \frac{1+\sqrt{z}}{1-\sqrt{z}} = \sum_{0}^{\infty} \frac{z^n}{2n+1}. \tag{1.14}$$

**Lemma 1.3.11.** *(Sugawa and Wang 2015 [86]) For $f, g \in \mathcal{K}(\frac{3}{5})$ we have $\dfrac{f+g}{2} \in \mathcal{S}^*$.*

**Lemma 1.3.12.** *[47] If $f, g \in \mathcal{K}$ then $\dfrac{fg}{z} \in \mathcal{S}^*$. This product is not starlike of any order greater than zero when $f(z) = g(z) = \dfrac{z}{1+z}$.*

**Definition 1.3.1.** *An analytic function $f(z)$ is called strongly convex of order $\beta$ with $(0 < \beta \leq 1)$ satisfying the inequality*

$$\left| \arg \left\{ 1 + z\frac{f''(z)}{f'(z)} \right\} \right| < \beta \frac{\pi}{2}$$

*for $z \in \mathbb{D}$ and the set of all such functions is denoted by $\mathcal{K}^*[\beta]$.*

$$f(z) \in \mathcal{K}[\beta] \Longleftrightarrow zf'(z) \in \mathcal{S}^*[\beta] \tag{1.15}$$

Nunokawa (1993) proved that for $0 < \beta < 1$, if $f(z) \in \mathcal{K}\big[\delta(\beta)\big]$ then $f(z) \in \mathcal{S}^*[\beta]$ where

$$\delta(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta n(\beta) \sin \frac{(1-\beta)\pi}{2}}{m(\beta) + \beta n(\beta) \cos \frac{(1-\beta)\pi}{2}}$$

while $m(\beta) = (1+\beta)^{\frac{1+\beta}{2}}$ and $n(\beta) = (1-\beta)^{\frac{\beta-1}{2}}$.

**Lemma 1.3.13.** *(Umezawa [88]) Let $f \in \mathcal{A}$ and $f'(z) \neq 0$ on $|z| = 1$. If there holds the relation $\displaystyle\int_0^{2\pi} \Big|\mathbf{Re}\,\big\{1 + z\frac{f''(z)}{f'(z)}\big\}\Big| d\theta < 4\pi$ for $|z| = 1$ then $f(z)$ is convex in one direction and hence $f(z)$ is univalent in $|z| \leq 1$.*

**Definition 1.3.2.** *A domain $D$ is called convex in the direction of $\alpha$ $(0 \leq \alpha < \pi)$ if it's intersection with every line parallel to the line pass through $0$ and $e^{i\alpha}$ is either empty or an interval.*

A univalent function $f$ in $D$ is said to be convex in the direction of $\alpha$ if $f(\mathbb{D})$ is convex in the direction of $\alpha$. We say that $f$ is convex in one direction if there exists an $\alpha$ such that $f$ is convex in the direction of $\alpha$.

**Lemma 1.3.14.** *(Pommerenke [61]) Let $f$ be an analytic function in $\mathbb{D}$, $f(0) = 0$, and $f'(0) \neq 0$, and let*

$$\phi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})}$$

*where $\theta \in \mathbb{R}$. If*

$$\mathbf{Re}\left\{\frac{zf'(z)}{\phi(z)}\right\} > 0 \quad , \quad z \in \mathbb{D}$$

*then $f$ is convex in the direction of the real axis.*

Lewandowski, Miller and Zotkiewicz, 1974 [**?**] have introduced the class of $\gamma$-starlike functions, denoted here by $\mathcal{G}(1, \gamma)$, which satisfy

$$\mathbf{Re}\left\{\Big(z\frac{f'(z)}{f(z)}\Big)^{1-\gamma}\Big(1 + z\frac{f''(z)}{f'(z)}\Big)^{\gamma}\right\} > 0 \quad , \quad z \in \mathbb{D}$$

Nunokawa and Sokol [56] extend this class to $\mathcal{G}(\alpha,\gamma)$, called $\gamma$-strongly starlike functions of order $\alpha$ consists of functions $f \in \mathcal{A}$ satisfying

$$\left|\arg\left\{\left(z\frac{f'(z)}{f(z)}\right)^{1-\gamma}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \alpha\frac{\pi}{2} \quad , \quad z \in \mathbb{D} \tag{1.16}$$

where $0<\alpha\le 1$, $\gamma>0$ and $f(z)$ is verified with $f(z)f'(z)\Big(1+z\dfrac{f''(z)}{f'(z)}\Big) \neq 0$ in $z \in \mathbb{D}-\{0\}$. Note that $\mathcal{G}(0,\beta) \subset \mathcal{S}^*[\beta]$,

**Theorem 1.3.2.** *Let $0<\alpha\le 1$, $\gamma>0$ and $f \in \mathcal{A}$ is of the form (1.1) satisfies (1.16). If the equation, with respect to $x$,*

$$x+\frac{2\gamma}{\pi}\tan^{-1}\frac{xn(x)\sin\frac{(1-x)\pi}{2}}{m(x)+xn(x)\cos\frac{(1-x)\pi}{2}} = \alpha \tag{1.17}$$

*while $m(x)=(1+x)^{\frac{1+x}{2}}$ and $n(x)=(1-x)^{\frac{x-1}{2}}$, has a solution $\beta \in (0,1]$, then $f \in \mathcal{G}(\alpha,\gamma)$.*

**Theorem 1.3.3.** *Let $0<\alpha\le 1$, $0<\gamma\le 1$ and $f \in \mathcal{A}$ is of the form (1.1) satisfies (1.16). If the equation (1.17) has a solution $0<\alpha_0\le 1$, then $f \in \mathcal{K}[\frac{(1-\gamma)\alpha_0+\alpha}{\gamma}]$.*

**Corollary 1.3.4.** *Assume that the equation (1.17) has a solution $0<\alpha_0<\alpha\le 1$. If $0<\delta<\gamma$, then $\mathcal{G}(\alpha,\gamma) \subset \mathcal{G}(\alpha,\delta)$.*

**Theorem 1.3.5.** *Let $0<\alpha\le 1$, $\gamma<0$ and $f \in \mathcal{A}$ is of the form (1.1) satisfies (1.16). If $\beta = \dfrac{\alpha-\gamma}{1-\gamma}$ then $f \in \mathcal{S}^*[\beta]$.*

Sim and Kwon [81] introduce this class (2013):

**Definition 1.3.3.** *For real numbers $\alpha$ and $\beta$ with $0\le\alpha<1<\beta$, a function $f(z) \in \mathcal{A}$ belonge to the class $\mathcal{K}(\alpha,\beta)$ if $f(z)$ satisfies this inequality*

$$\alpha < \mathbf{Re}\left(1+z\frac{f''(z)}{f'(z)}\right) < \beta \quad , \qquad z \in \mathbb{D} \tag{1.18}$$

Clearly $\mathcal{K}(\alpha,\beta) \subset \mathcal{K}$ and (1.22) gives these conditions equivalently:

$$1+z\frac{f''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad , \quad z \in \mathbb{D}\,,\ 0\le\alpha<1 \tag{1.19}$$

$$1+z\frac{f''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z} \quad , \quad z \in \mathbb{D}\,,\ \beta>1 \tag{1.20}$$

They used the function

$$p(z) = 1 + i\frac{\beta-\alpha}{\pi}\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right) \quad , \quad z \in \mathbb{D} \tag{1.21}$$

which maps $\mathbb{D}$ on convex strip $\alpha < \mathbf{Re}\, w < \beta$ was introduced by Kuroki and Owa [43], so

**Lemma 1.3.15.** *For real numbers $\alpha$ and $\beta$ with $0 \le \alpha < 1 < \beta$, a function $f(z) \in \mathcal{A}$ belonge to the class $\mathcal{K}(\alpha,\beta)$ iff*

$$1 + z\frac{f''(z)}{f'(z)} \prec 1 + i\frac{\beta-\alpha}{\pi}\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right) \quad , \quad z \in \mathbb{D} \tag{1.22}$$

and obtained coefficient estimates bounds for $f \in \mathcal{K}(\alpha,\beta)$ as following

$$|a_n| \le \begin{cases} \frac{1}{2}|B_1| & ; \quad n = 2, \\ \frac{|B_1|}{n(n-1)}\prod_{k=1}^{n-2}\left(1 + \frac{|B_1|}{k}\right) & ; \quad n = 3, 4, \cdots . \end{cases} \tag{1.23}$$

where

$$|B_1| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{(1-\alpha)\pi}{\beta-\alpha}.$$

**Lemma 1.3.16.** *For real numbers $\alpha$ and $\beta$ with $0 \le \alpha < 1 < \beta < 2$, if $f \in \mathcal{K}(\alpha,\beta)$ then*

$$\frac{1}{2-\alpha} < \mathbf{Re}\sqrt{f'(z)} < \frac{1}{2-\beta}$$

## 1.4 Close-to-Convex

A domain $D$ is said to be close-to-convex, if the complement of $D$ can be written as a union of non-crossing half lines. An univalent function $f$ in $\mathbb{D}$ is said to be close-to-convex if its range $f(\mathbb{D})$ is a close-to-convex domain.

**Definition 1.4.1.** *A function of the form (1.1) is called close-to-convex if there is a starlike function $g$ and a function $h \in \mathbb{P}$ such that*

$$zf'(z) = g(z)h(z) \quad , \quad z \in \mathbb{D}$$

The close-to-convex subclass of $\mathcal{S}$, include all close-to-convex univalent function, denote by $\mathcal{C}$. This class was introduced by Kaplan (1952) and shown by him to consist of univalent functions. It is clear that $\mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}$.

The equivalent definition is that a function $f(z)$ is called close-to-convex if there is a starlike function $g$ such that

$$\mathbf{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \quad , \quad z \in \mathbb{D} \tag{1.24}$$

This is similar to say there is a (not necessarily normalized) convex univalent function $h$ such that

$$\mathbf{Re}\left\{\frac{f'(z)}{h'(z)}\right\} > 0 \quad , \quad z \in \mathbb{D} \tag{1.25}$$

Using Noshiro-Warschawski theorem ([24], p.47), every function $f(z) \in \mathcal{A}$ with $\mathbf{Re}\, f'(z) > 0$ is univalent and close-to-convex. With suitable choose of $g$ in (1.24), one obtains a subclass of close-to-convex functions

$$\mathbf{Re}\left\{\prod_{j=1}^{n}(z - e^{i\alpha_j})^{\sigma_j} f'(z)\right\} > 0 \tag{1.26}$$

where $0 \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_n \leq 2\pi$, $0 \leq \sigma_j \leq 1$ and $\displaystyle\sum_{j=1}^{n} \sigma_j \leq 2$.

**Lemma 1.4.1.** *(Kaplan [24]) Let $f$ be analytic and locally univalent in $\mathbb{D}$ and satisfy the following condition*

$$\mathbf{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > -\frac{1}{2}$$

*for all $z \in \mathbb{D}$, then $f$ is univalent and close-to-convex in $\mathbb{D}$.*

**Definition 1.4.2.** *A function $f(z) \in \mathcal{A}$ is called close-to-convex of order $\alpha$ $(0 \leq \alpha < 1)$ if satisfying the condition*

$$\mathbf{Re}\left\{\frac{f'(z)}{h'(z)}\right\} > \alpha \quad , \quad z \in \mathbb{D}$$

*for some (not necessarily normalized) convex univalent function $h$ on $\mathbb{D}$.* The class of all such functions is denoted by $\mathcal{C}(\alpha)$.

## 1.5 $\mathbb{P}$ class

This set is the family of all functions $\mathfrak{p}$ analytic in $\mathbb{D}$ for which $\mathbf{Re}\,\mathfrak{p}(z) > 0$, $\mathfrak{p}(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{D}$. A Herglotz representation formula shows that

**Lemma 1.5.1.** *$\mathfrak{p}(z) \in \mathbb{P}$ iff $p(z) = \displaystyle\int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z} d\gamma(t)$ such that $\gamma$ is increasing and $\gamma(2\pi) - \gamma(0) = 1$.*

**Lemma 1.5.2.** *(Caratheodory Lemma [62]) If $\mathfrak{p}(z) \in \mathbb{P}$, then $|c_n| \leq 2$ for each $n$.*

It's trivial that

$$f(z) \in \mathcal{S}^* \Leftrightarrow z\frac{f'(z)}{f(z)} \in \mathbb{P}$$

$p_0 = \dfrac{1+z}{1-z}$ is the extremal function for this class. that is starlike and convex.

## 1.6 Bazilevič

This class is a generalization for close-to-convex class. Let $g(z)$ is starlike in $\mathbb{D}$, $\mathfrak{p}$ is analytic function with $\mathbf{Re}\ \mathfrak{p}(z) > 0$ in $\mathbb{D}$ and $a > 0$, then the function

$$f(z) = \Big( \int_0^z \mathfrak{p}(z) g(z)^\alpha \zeta^{i\beta - 1} d\zeta \Big)^{\frac{1}{\alpha + i\beta}} \qquad \alpha > 0,\ \beta \in \mathbb{R} \tag{1.27}$$

has been shown by Bazilevič [10] to be a analytic and univalent function in $\mathbb{D}$. This class of functions denote by $B(\alpha + i\beta)$. Note that $B(1)$ is the class of normalized close-to-convex functions. In [25] has been showed that if $f(z) \in B(\alpha + i\beta)$ with $\mathfrak{p}(z) = 1$ then $f(z)$ must satisfy

$$\mathbf{Re}\,\{(\alpha - 1 + i\beta) z\frac{f'(z)}{f(z)} + (1 + z\frac{f''(z)}{f'(z)})\} > 0 \tag{1.28}$$

for $z \in \mathbb{D}$. Conversely if $f(z)$ is analytic $\mathbb{D}$ with $f(0) = 0$, $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and satisfy in (1.28) then $f(z)$ can be written in the form (1.27), with $\mathfrak{p}(z) = 1$.

## 1.7 Spirallike Class

A generalization of starlikeness leads us to a useful property known as spirallikeness introduced by Spacek in 1933. A logarithmic spiral is a curve in the complex plane of the form $w = w_0 e^{-\lambda t}$ for $t \in \mathbb{R}$, where $w_0 \neq 0$ and $\lambda$ are complex constants with $\mathbf{Re}\,\lambda \neq 0$. With no loss of generality in assuming $\lambda = e^{i\alpha}$ with $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then the curve is called an $\alpha$-spiral.

A domain $D$ containing the origin is said to be $\alpha$-spirallike if for each point $w_0 \neq 0$ in $D$, the arc of the $\alpha$-spiral from $w_0$ to the origin lies entirely in $D$. An $\alpha$-spirallike domain is simply connected.

A function $f \in \mathcal{A}$ is said to be $\alpha$-spirallike if it's range is $\alpha$-spirallike. A function is spirallike if it is $\alpha$-spirallike for some $\alpha$. The 0-spirallike functions are starlike functions. Spirallikeness can be characterized by an analytic condition which is a slight generalization of the condition for starlikeness:

**Theorem 1.7.1.** *[24] Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ for $z \in \mathbb{D} - \{0\}$ and $\alpha \in [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$. Then $f$ is $\alpha$-spirallike if and only if*

$$\mathbf{Re}\left\{ e^{-i\alpha} z \frac{f'(z)}{f(z)} \right\} > 0 \tag{1.29}$$

The class of all $\alpha$-spirallike functions in $\mathcal{S}$ is shown by $\mathcal{S}^{sp}(\alpha)$. The Theorem asserts (1.29) is a sufficient condition for univalence. Geometric considerations show that for $\alpha \neq 0$, an $\alpha$-spirallike function need not be close-to-convex. An example is the function

$$f(z) = \frac{z}{(1-z)^{2e^{i\alpha}\cos\alpha}} \in \mathcal{S}^{sp}(\alpha) \tag{1.30}$$

which maps $\mathbb{D}$ onto the complement of an arc of an $\alpha$-spiral. This function plays the role of the Koebe function in extremal problems for $\alpha$-spirallike functions. On the other hand, a close-to-convex function need not be spirallike. An example is the function

$$f(z) = \frac{z - z^2\cos\phi}{(1 - e^{i\phi}z)^2}\,, \quad \cos\phi \neq 0 \tag{1.31}$$

which maps $\mathbb{D}$ onto the complement of a nonradial half-line. Kulshrestha [41] introduced a subclass $\mathcal{S}^{sp}(\alpha, \gamma) \subset \mathcal{S}^{sp}(\alpha)$ of $\gamma$-spiral functions of order $\alpha$ as follows:

**Definition 1.7.1.** *[41] Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ for $z \in \mathbb{D} - \{0\}$. Then $f \in \mathcal{S}^{sp}(\alpha, \gamma)$ if and only if there exist real numbers $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $0 \leq \gamma < 1$, such that*

$$\mathbf{Re}\left\{e^{i\alpha} z \frac{f'(z)}{f(z)}\right\} > \gamma \cos\alpha\,, \quad z \in \mathbb{D} \tag{1.32}$$

## 1.8 $\mathcal{T}$ Class

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an analytic and univalent function $f$ is in $\mathcal{T}$ if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$$

**Lemma 1.8.1.** *[80] The function $f = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $\mathcal{T}^*(\alpha)$, if $\sum_{n=2}^{\infty} n|a_n| \leq 1$.*

We also denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{TK}(\alpha)$ the subclasses of $\mathcal{T}$ that are, respectively, starlike of order $\alpha$ and convex of order $\alpha$. $\mathcal{T} = \mathcal{T}^*(0)$

**Lemma 1.8.2.** *[80] The function $f = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{A}$, is in $\mathcal{T}^*(\alpha)$, where $0 \leq \alpha < 1$, if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \leq 1 \tag{1.33}$$

**Lemma 1.8.3.** *[80] If the function $f = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{T}^*(\alpha)$, with $0 \leq \alpha < 1$, then $|a_n| \leq \dfrac{1-\alpha}{n-\alpha}$ with equality only for functions of the form $f(z) = z - \dfrac{1-\alpha}{n-\alpha} z^n$.*

**Lemma 1.8.4.** *[80] The function $f = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $\mathcal{TK}^*(\alpha)$, where $0 \leq \alpha < 1$, iff*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| \leq 1 \tag{1.34}$$

**Lemma 1.8.5.** *[80] If the function $f \in \mathcal{T}^*(\alpha)$, with $0 \le \alpha < 1$, then*

$$r - \frac{1-\alpha}{2-\alpha} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2-\alpha} r^2 \tag{1.35}$$

$$1 - \frac{2(1-\alpha)}{2-\alpha} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{2-\alpha} r \tag{1.36}$$

*with equality for function $f(z) = z - \frac{1-\alpha}{2-\alpha} z^2$ when $z = \pm r$.*

**Lemma 1.8.6.** *[80] If the function $f \in \mathcal{TK}^*(\alpha)$, with $0 \le \alpha < 1$, then*

$$r - \frac{1-\alpha}{2(2-\alpha)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2(2-\alpha)} r^2 \tag{1.37}$$

$$1 - \frac{1-\alpha}{2-\alpha} r \le |f'(z)| \le 1 + \frac{1-\alpha}{2-\alpha} r \tag{1.38}$$

*with equality for function $f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2$ when $z = \pm r$.*

**Lemma 1.8.7.** *[80] If the function $f \in \mathcal{TK}^*(\alpha)$, then $f \in \mathcal{T}^*(\frac{2}{3-\alpha})$,The result is sharp with $f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2$ being extremal.*

**Lemma 1.8.8.** *[80] If the function $f \in \mathcal{T}^*(\alpha)$, then convexity radius of $f(z)$ is $r_{con}(\alpha) = \inf_{n \ge 2} \left( \frac{n-\alpha}{n^2(1-\alpha)} \right)^{\frac{1}{n-1}}$. The result is sharp with $f_n(z) = z - \frac{1-\alpha}{n-\alpha} z^n$ being extremal for some $n$.*

**Lemma 1.8.9.** *[80] Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$, then $\sum_{n=2}^{\infty} n|a_n| \le 1$.*

## 1.9 Starlike with respect to symmetric points

Let $f(z)$ be analytic in $\mathbb{D}$ and suppose that for every $r$ less than and sufficiently close to one and every $\zeta$ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as $z$ traverses the circle $|z| = r$ in the positive direction, that is to say

$$\mathbf{Re}\, \frac{zf'(z)}{f(z) - f(-\zeta)} > 0\ , \quad z = \zeta, |\zeta| = r \tag{1.39}$$

Then $f(z)$ is said to be starlike with respect to symmetrical points. Obviously the class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions and odd functions starlike with respect to the origin.

This is equivalent with following expression:

**Definition 1.9.1.** *[76] A function $f \in \mathcal{A}$ is starlike with respect to symmetric points if*

$$\mathbf{Re}\,\frac{2zf'(z)}{f(z)-f(-z)} > 0\ , \quad z \in \mathbb{D} \tag{1.40}$$

*These functions are also univalent and the class of all such functions is denoted by $\mathcal{S}_s^*$.*

**Lemma 1.9.1.** *[76] Let $f \in \mathcal{A}$ be univalent and starlike with respect to symmetrical points in $\mathbb{D}$. Then $|a_n| \le 1$ for $n \ge 2$, equality being attained by the function $\dfrac{z}{1+\varepsilon z}$, $|\varepsilon| = 1$.*

also a generalized condition is as follows:

**Lemma 1.9.2.** *[76] Let $f \in \mathcal{A}$, and suppose that for a positive integer $k$ there holds the inequality*

$$\mathbf{Re}\,\frac{zf'(z)}{\sum_{j=0}^{k-1}\frac{f(\varepsilon^j z)}{\varepsilon^j}} > 0\ , \quad z \in \mathbb{D} \tag{1.41}$$

*where $\varepsilon = e^{2\pi i/k}$. Then $f(z)$ is univalent and close-to-convex in $\mathbb{D}$.*

**Definition 1.9.2.** *A function $f(z) \in \mathcal{A}$ is starlike with respect to symmetric points of order $\alpha$ if*

$$\mathbf{Re}\,\frac{2zf'(z)}{f(z)-f(-z)} > \alpha\ , \quad z \in \mathbb{D}$$

*where $0 \le \alpha < 1$. These functions are also univalent and the class of all such functions is denoted by $\mathcal{S}_s^*(\alpha)$.*

## 1.10 $\mathcal{UST}$ class

As we said, starlikeness is a hereditary property for conformal mappings, but it is not always true that $f \in \mathcal{S}^*$ maps each disk $|z - z_0| < \rho < 1 - |z_0|$ onto a domain starlike with respect to $f(z_0)$. This matter proved by Brown in 1989 [15], He showed that for each $f \in \mathcal{S}$ and for all sufficiently small disks in $\mathbb{D}$, the image of this small disk is starlike. The set of such functions with this property that maps each disk $\{|z - z_0| < \rho\} \subset \mathbb{D}$ onto a domain starlike with respect to $f(z_0)$ introduced by Goodman [32] and studied in analytic and geometric view.

**Definition 1.10.1.** *A function $f(z) \in \mathcal{S}^*$ is said to be uniformly starlike in $\mathbb{D}$ if it has the property that for every circular arc $\gamma$ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ be starlike with respect to $f(\zeta)$. We denote the family of all uniformly starlike functions by $\mathcal{UST}$ and we have [32],*

$$\mathcal{UST} = \left\{ f(z) \in \mathcal{S} : \mathbf{Re}\, \frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > 0\ , \quad (z,\zeta) \in \mathbb{D}^2 \right\} \tag{1.42}$$

We may define $\dfrac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} = 1$ and it's clear that $\mathcal{UST} \subset \mathcal{S}^*$. This class is preserved under rotations, $e^{-i\alpha} f(e^{i\alpha} z)$ for $\alpha \in \mathbb{R}$ and transformations $\dfrac{1}{t} f(tz)$, $0 < t \leq 1$ are preserve this class also. $\mathcal{UST}$ class isn't linearly invariant, in fact the disk automorphism of the function $f(z) = \dfrac{z}{1 - \frac{1}{2}z}$ is not belong to this class.

For $\zeta = -z$ in (1.42), evidently $\mathcal{UST} \subset \mathcal{S}_s^*$ and hence for $f \in \mathcal{UST}$, $|a_n| \leq 1$ but there is a better bound $|a_n| \leq \dfrac{2}{n}$ proved by Charles Horowitz, mentioned in [32]. Determination of the sharp coefficient estimates for $\mathcal{UST}$ class is an open problem. The most of properties of the class $\mathcal{UST}$ are difficult to establish. Goodman showed that

$$f(z) = \frac{z}{1 - Az} \in \mathcal{UST} iff |A| \leq \frac{\sqrt{2}}{2} \tag{1.43}$$

He also proved for $|B| \leq \dfrac{n}{\sqrt{2}}$, the function $f(z) = z + Bz^n$ belongs to $\mathcal{UST}$ where $n > 1$. Merkes and Salmassi [49] improved this bound to be $|B| \leq \sqrt{\dfrac{n+1}{2n^3}}$ for $n > 1$, besides this bound need not be sharp. The sharp upper bound was obtained by Nezhmetdinov in 1997 ([55], Corollary 4, p. 47) and shows for $n = 2$, $|B| \leq \dfrac{1}{2.31}$ and for $n = 3$, $|B| \leq \dfrac{1}{3.573}$. Rønning [**?**], Merkes and Salmassi [49] showed the following important results:

**Lemma 1.10.1.** *([69], Lemma 3.3, p.236)* $f(z) \in \mathcal{UST}$ *iff for every* $z \in \mathbb{D}$, $|x| = 1$, $\mathbf{Re}\, \dfrac{f(z) - f(xz)}{(1-x)zf'(z)} \geq 0$.

**Lemma 1.10.2.** *[?]* $f(z) \in \mathcal{UST}$ *iff for every* $z \in \mathbb{D}$, $|t| = 1$, $\mathbf{Re}\, \dfrac{(1-t)zf'(z)}{f(z) - f(tz)} > 0$.

**Lemma 1.10.3.** *([49], Theorem 4, p.451)* $f \in \mathcal{UST}$ *if for all* $z, w \in \mathbb{D}$,

$$\mathbf{Re}\, \frac{f'(w)}{f'(z)} > 0$$

*and if* $f \in \mathcal{UST}$ *then for all* $z, w \in \mathbb{D}$,

$$\mathbf{Re}\left(\frac{f'(w)}{f'(z)}\right)^{\frac{1}{2}} > 0$$

*The exponent* $\frac{1}{2}$ *is the best possible.*

Further investigations of $\mathcal{UST}$ class obtained by Taylor series exansion of (1.42) about $z$ and $\zeta$ seperately. Let $p(z) = p_0 + p_1 z + p_2 z^2 + \cdots$, $q(z) = q_0 + q_1 z + q_2 z^2 + \cdots$ and

$$\frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)} = \sum_{n=0}^{\infty} p_n(\zeta) z^n = \sum_{n=0}^{\infty} q_n(z) \zeta^n \ , \quad (z, \zeta) \in \mathbb{D}^2 \tag{1.44}$$

such that $\mathbf{Re}\, p(z) > 0$ and $\mathbf{Re}\, q(z) > 0$,

**Lemma 1.10.4.** *([32], Lemma 1, p.365) If* $f \in \mathcal{UST}$, *then*

$$p_0(\zeta) = \frac{f(\zeta)}{\zeta}\ , \ p_1(\zeta) = \frac{f(\zeta)[1 - 2a_2\zeta] - \zeta}{\zeta^2}\ , \ q_0(z) = \frac{f(z)}{zf'(z)}\ , \ q_1(z) = \frac{f(z) - z}{z^2 f'(z)}$$

*and*

$$|p_1(\zeta)| \leq 2\mathbf{Re}\, p_0(\zeta)\ , \ |q_1(z)| \leq 2\mathbf{Re}\, q_0(z)$$

This lemma and coefficients bound estimates $|a_n| \leq \dfrac{2}{n}$ lead us to growth inequality for $f \in \mathcal{UST}$,

$$\frac{r}{1+2r} \leq |f(z)| \leq -r + 2\ln \frac{1}{1-r}$$

for $|z| = r < 1$. Finally this shows the Koebe constant for the family $\mathcal{UST}$, is

$$\frac{1}{3} \leq K(\mathcal{UST}) \leq 1 - \frac{\sqrt{3}}{4}$$

To obtain a sufficient condition which concludes a function belongs to $\mathcal{UST}$, we need convolution method. For determining the greatest value of $\delta$ such that the condition

$$\sum_{n=2}^{\infty} n|a_n| \leq \delta$$

implies that $f \in \mathcal{UST}$, Goodman showed that $\delta = \dfrac{\sqrt{2}}{2} = 0.7071\ldots$ is an acceptable value but the sharp value for $\delta$ must not exceed $\dfrac{\sqrt{3}}{2} = 0.8660\ldots$. Finally

**Lemma 1.10.5.** *(Nezhmetdinov [55]) If $f \in \mathcal{A}$ satisfies the condition $\sum_{n=2}^{\infty} n|a_n| \leq \delta_0$ then $f \in \mathcal{UST}$. The constant $\delta_0$ on the right-hand side is equal to $\delta_0 = \dfrac{1}{\sqrt{M}} = 0.7963\ldots$ that $M$ is the best possible mentioned in lemma 1.14.3.*

**Lemma 1.10.6.** *([49], Theorem 1, p.450) Let $f \in \mathcal{A}$. $f \in \mathcal{UST}$ iff for all $\alpha, \beta \in \overline{\mathbb{D}}$,*

$$\mathbf{Re}\, \frac{f(z) * \frac{z}{(1-\alpha z)(1-\beta z)}}{f(z) * \frac{z}{(1-\alpha z)^2}} \geq 0\ , \quad z \in \mathbb{D}$$

**Open Problem 1.** *Ronning [?] proved that $\mathcal{UST} \not\subseteq \mathcal{S}^*(\frac{1}{2})$ and posed the problem of determining the largest $\alpha$ such that $\mathcal{UST} \subset \mathcal{S}^*(\alpha)$. Nezhmetdinov [?] showed that $\mathcal{UST} \not\subseteq \mathcal{S}^*(\alpha_0)$ for some $\alpha_0 \approx 0.1483$. Determine the largest $\alpha$ such that $\mathcal{UST} \subset \mathcal{S}^*(\alpha)$.*

the map $\ell(z) = \dfrac{z}{1-z} = \dfrac{1}{2}(p_0 - 1)$ is not in $\mathcal{UST}$.

## 1.11 $\mathcal{UCV}$ class

As we said, starlikeness is a hereditary property for conformal mappings, but it is not always true that $f \in \mathcal{S}^*$ maps each disk $|z - z_0| < \rho < 1 - |z_0|$ onto a domain starlike with respect to $f(z_0)$. This matter proved by Brown in 1989 [15], He showed that for each $f \in \mathcal{S}$ and for all sufficiently small disks in $\mathbb{D}$, the image of this small disk is starlike. The set of such functions with this property that maps each disk $\{|z - z_0| < \rho\} \subset \mathbb{D}$ onto a domain starlike with respect to $f(z_0)$ introduced by Goodman [32] and studied in analytic and geometric view.

**Definition 1.11.1.** *A function $f(z) \in \mathcal{S}$ is said to be uniformly convex in $\mathbb{D}$ if it has the property that for every circular arc $\gamma$ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ be convex. We denote the family of all uniformly convex functions by $\mathcal{UCV}$ and we have [31],*

$$\mathcal{UCV} = \left\{ f(z) \in \mathcal{S} : \mathbf{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0\ , \quad (z,\zeta) \in \mathbb{D}^2 \right\} \tag{1.45}$$

It's clear that $\mathcal{UCV} \subset \mathcal{K}$. For $\zeta = -z$ in (1.45), evidently $\mathcal{UCV} \subset \mathcal{K}(\frac{1}{2})$ and hence for $f \in \mathcal{UCV}$, $|a_n| \leq \dfrac{1}{n}$. The class $\mathcal{UCV}$ isn't a linear-invariant family, this was proved by Goodman ([31], Theorem 5, p.90) with function $f(z) = \dfrac{z}{1-Az}$. He showed $f(z) = \dfrac{z}{1-Az} \in \mathcal{UCV}$ iff $|A| \leq \dfrac{1}{3}$.

**Lemma 1.11.1.** *(Nezhmetdinov [55]) For $n \geq 2$, $f = z + a_n z^n \in \mathcal{UCV}$ iff $|a_n| \leq \dfrac{1}{n(2n-1)}$.*

**Lemma 1.11.2.** *(Nezhmetdinov [55]) If $f \in \mathcal{A}$ satisfies the condition $\sum_{n=2}^{\infty} n(2n-1)|a_n| \leq 1$ then $f \in \mathcal{UCV}$. The constant 1 on the right-hand side is the best possible.*

There is a significant one-variable characterization of $\mathcal{UCV}$ found by Ronning ([71], Theorem 1, p.190) and Ma and Minda [?, Theorem 2, p.162] independently:

**Lemma 1.11.3.** *$f \in \mathcal{UCV}$ iff*

$$\mathbf{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > \left| z\frac{f''(z)}{f'(z)} \right|\ , \quad z \in \mathbb{D} \tag{1.46}$$

Futhermore if $\left| z\dfrac{f''(z)}{f'(z)} \right| < \dfrac{1}{2}$ then $f \in \mathcal{UCV}$.

## 1.12 $\mathcal{S}_{\mathcal{P}}$ class

Now, let $w = 1 + z\dfrac{f''(z)}{f'(z)}$, according to (1.46) define

$$\Omega_p = \{w \in \mathbb{C} : \mathbf{Re}\, w > |w-1|\,\} \tag{1.47}$$

The set $\Omega_p$ is the interior of the parabola $(\mathbf{Im}\, w)^2 = 2\mathbf{Re}\ w - 1$ which it is symmetric with respect to the real axis and has $(\dfrac{1}{2}, 0)$ as its vertex. So, $f \in \mathcal{UCV}$ iff $1 + z\dfrac{f''(z)}{f'(z)} \in \Omega_p$. The class $\mathcal{S}_{\mathcal{P}}$ defined in the following way:

**Definition 1.12.1.** *The class $\mathcal{S}_{\mathcal{P}}$ of parabolic starlike functions consists of all functions $f \in \mathcal{A}$ satisfying condition*

$$\mathbf{Re}\left(z\frac{f'(z)}{f(z)}\right) > \left|z\frac{f'(z)}{f(z)} - 1\right|\ , \quad z \in \mathbb{D} \tag{1.48}$$

It's clear that $\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}^*$ and since the parabolic region $\Omega_p$ is contained in the halfplane $\{w : \mathbf{Re}\, w > \dfrac{1}{2}\}$ and the sector $\{w : |\arg w| < \dfrac{\pi}{4}\}$, so $\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}^*(\frac{1}{2}) \cap \mathcal{S}^*_{\frac{1}{2}}$ [71]. Also (1.46) and (1.48) show that

$$f \in \mathcal{UCV} \Longleftrightarrow zf'(z) \in \mathcal{S}_{\mathcal{P}} \tag{1.49}$$

by Alexander theorem (1.3.1). Is there such relation like (1.49) between $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{UST}$ classes? The answer is no. Goodman [31] and Ronning [69] show

$$\mathcal{S}_{\mathcal{P}} \nsubseteq \mathcal{UST}\ \ ,\ \ \mathcal{UST} \nsubseteq \mathcal{S}_{\mathcal{P}}$$

Futhermore if $\left|z\dfrac{f'(z)}{f(z)} - 1\right| < \dfrac{1}{2}$ then $f \in \mathcal{S}_{\mathcal{P}}$.

## 1.13 Ma-Minda Starlike

Ma and Minda [1992] gave a unified presentation of some subclasses of starlike and convex functions by subordination. Let $\phi$ be an analytic function with positive real part and normalized by the conditions $\phi)0) = 1$, $\phi'(0) > 0$ and $\phi$ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. They introduced these subclasses for starlike and convex function class.

$$\mathcal{S}^*(\phi) = \{f \in A : z\frac{f'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{D}\}$$

In literature, the functions belonging to these classes are called Ma-Minda starlike and Ma-Minda convex, respectively.

## 1.14 Convolution over analytic functions

The convolution or Hadamard product of two functions $f(z)$ and $F(z)$ with power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is denoted by $f * F$ and is defined as[3]

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n, \tag{1.50}$$

The right half-plane mapping $\ell(z) = \dfrac{z}{1-z}$ acts as the convolution identity and the map $k(z) = \dfrac{z}{(1-z)^2}$ acts as derivative operation over function, also.

---

[3] The term *convolution* is used since

$$(f * F)(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} f\Big(\frac{z}{\zeta}\Big)F(z)\frac{d\zeta}{\zeta}$$

for $|z| < \rho < 1$.

We have some properties for convolution over analytic functions f, F

$$
\begin{aligned}
& f * F = F * f \\
& \alpha(f * F) = \alpha f * F \\
& f * l = f \\
& zf * zF = z(f * F) \\
& f * \frac{1}{\alpha} l(\alpha z) = \frac{1}{\alpha} f(\alpha z) \\
& \overline{f * F} = \overline{f} * \overline{F} \\
& f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3 \\
& \alpha(f * F) = \alpha f * F \\
& zf'(\alpha z) = f * \frac{z}{(1-\alpha z)^2} \\
& \frac{1}{\alpha} f(\alpha z) = f * \frac{z}{1-\alpha z} \\
& zf'(z) = f * k(z) \\
& z(f * F)' = zf' * F = f * zF' \\
& \frac{f(\alpha z) - f(\beta z)}{\alpha - \beta} = f * \frac{z}{(1-\alpha z)(1-\beta z)}
\end{aligned}
$$

where $\alpha \in \mathbb{C}$,

For real-value $g$ function we have

$$\mathbf{Re}\,(f * g) = \mathbf{Re}\, f * g \quad , \quad \mathbf{Im}(f * g) = \mathbf{Im} f * g$$

In case of analytic functions $f(z)$ and $F(z)$, this subject is significant and has investigated by some people. The classes of starlike, convex and close-to-convex functions are closed under convolution with convex functions. This was conjectured by Pólya and Schoenberg [60] and proved by Ruscheweyh and Shiel-Small [75] in 1973:

(a) If $f \in \mathcal{K}$ and $F \in \mathcal{K}$, then $f * F$ also belongs to $\mathcal{K}$,

(b) If $f \in \mathcal{K}$ and $F \in \mathcal{C}$, then $f * F$ also belongs to $\mathcal{C}$,

(c) If $f \in \mathcal{K}$, $F \in \mathcal{K}$ and $G \in \mathcal{C}$, Then $\dfrac{f * zG'}{f * zF'}$ takes all its values in a convex domain $D$ if $\dfrac{G'}{F'}$ takes all its values in $D$.

Futhermore,

(d) If $f \in \mathcal{C}$ and $F \in \mathcal{S}^*$, then $f * F$ also belongs to $\mathcal{S}^*$.

| | | |
|---|---|---|
| If $\sum_{n=2}^{\infty} n^2\|a_n\| \leq 1$ | Then $f \in \mathcal{C}$ and if $f \in \mathcal{C}$ then | $\|a_n\| \leq 1$ |
| If $\sum_{n=2}^{\infty} n\|a_n\| \leq 1$ | Then $f \in \mathcal{S}^*$ and if $f \in \mathcal{S}^*$ then | $\|a_n\| \leq n$ |
| If $\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}\|a_n\| \leq 1$ | Then $f \in \mathcal{S}^*(\alpha)$ and if $f \in \mathcal{S}^*$ then | $\|a_n\| \leq n$ |
| If $\sum_{n=2}^{\infty} n^2\|a_n\| \leq 1$ | Then $f \in \mathcal{K}$ | |

**Lemma 1.14.1.** *(Ruscheweyh & Sheil-Small [75]) Let $f(z)$ and $F(z)$ be analytic in $\mathbb{D}$ with $f(0) = F(0) = 0$. If $f$ be convex and $F$ is starlike, then for each function $p(z)$, analytic in $\mathbb{D}$ and satisfying* **Re** *$p(z) > 0$, we have*

$$\mathbf{Re}\, \frac{(f * pF)(z)}{(f * F)(z)} > 0 \quad , \quad z \in \mathbb{D}$$

**Open Problem 2.** *Determine whether the class $\mathcal{UST}$ is closed under convolution with convex functions.*

For a given subset $\mathcal{V} \subset \mathcal{A}$, it's dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{f * g(z)}{z} \neq 0,\ \forall f \in \mathcal{V},\ \forall z \in \mathbb{D} \right\}$$

Nezhmetdinov (1997) proved that classes $\mathcal{UST}$ and $\mathcal{UCV}$ are dual sets for certain families of functions from $\mathcal{A}$, and showed ([55], Theorem 2, p.43) that the the dual set of the class $\mathcal{UST}$ is the subset of $\mathcal{A}$ consisting of functions $h : \mathbb{D} \to \mathbb{C}$ given by

$$h(z) = \frac{z\left(1 - \frac{w+i\alpha}{1+i\alpha} z\right)}{(1 - wz)(1 - z)^2}$$

where $\alpha \in \mathbb{R}$, $w \in \mathbb{C}$ and $|w| = 1$. He determined the uniform estimate $|a_n(h)| \leq dn$ for the $n$-th Taylor coefficient of $h$ in the dual set of $\mathcal{UST}$ with a sharp constant $d = \sqrt{M} \approx 1.2557$, where $M \approx 1.5770$ is the maximum value of a certain trigonometric expression. Using this, he showed that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{\sqrt{M}} \Longrightarrow f \in \mathcal{UST}$$

The bound $\dfrac{1}{\sqrt{M}}$ is sharp.

**Lemma 1.14.2.** *Let* $G_0 = \Big\{g \in \mathcal{A} : g(z) = \dfrac{z}{(1-z)^2}\Big[1 - \dfrac{i\alpha}{1+i\alpha}z\Big],\ \alpha \in \mathbb{R}\Big\}$, *then* $\mathcal{S}^* = G_0^*$ *and* $|a_n| \le n(2n-1)$ *for all* $g \in G_2$.

Nezhmetdinov [55] obtained a dual set for classes $\mathcal{UST}$ and $\mathcal{UCV}$:

**Lemma 1.14.3.** *Let* $G_1 = \Big\{g \in \mathcal{A} : g(z) = \dfrac{z}{(1-z)^2}\Big[1 - \dfrac{(t+i\alpha)}{1+i\alpha}z\Big]\Big[\dfrac{1}{1-tz}\Big],\ \alpha \in \mathbb{R},\ |t| = 1\Big\}$, *then* $\mathcal{UST} = G_1^*$ *and* $c_n = \sup\limits_{g \in G_1} |a_n| \le dn$ *for all* $n \ge 2$, *with the sharp constant* $d = \sqrt{M} = 1.2557\ldots$ *where* $M = S(\theta_0) = 1.5770\ldots$ *is the maxmimal value of*

$$S(\theta) = \frac{1}{2}\Big[1 + (\frac{\sin\theta}{\theta})^2 + \sqrt{\Big(1 + (\frac{\sin\theta}{\theta})^2\Big)^2 - (\frac{\sin 2\theta}{\theta})^2}\Big] \tag{1.51}$$

*on* $0 \le \theta \le \pi$. *Here the extremal point* $\theta_0 = 0.9958\ldots$ *is the unique solution of the equation*

$$\theta^3(\cos\theta + \cos 3\theta) - \theta^2 \sin 3\theta + \sin^3\theta = 0 \tag{1.52}$$

*on the segment* $0.8 \le \theta \le 1.3$.

**Lemma 1.14.4.** *Let* $G_2 = \Big\{g \in \mathcal{A} : g(z) = \dfrac{z}{(1-z)^3}\Big[1 - z - \dfrac{4z}{(\alpha+i)^2}\Big],\ \alpha \in \mathbb{R}\Big\}$, *then* $\mathcal{UCV} = G_2^*$ *and* $|a_n| \le n(2n-1)$ *for all* $g \in G_2$.

## 1.15 Prestarlike Function

**Definition 1.15.1.** *A function* $f(z) \in \mathbf{H}(\mathbb{D})$ *is called prestarlike of order* $\alpha$ *(with* $\alpha \le 1$*) if*

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in \mathcal{S}^*(\alpha)$$

*and for* $\alpha = 1$, $\mathbf{Re}\,\dfrac{f(z)}{z} > \dfrac{1}{2}$.

The set of all such functions is denoted by $R_\alpha$. Note that $R_0 = \mathcal{K}$ and $R_{\frac{1}{2}} = \mathcal{S}^*(\frac{1}{2})$.

Let $\Omega^* = \mathcal{C} - [1, \infty)$ and $f(z) \in \mathbf{H}(\Omega^*)$ we defined

$$(D^\beta f)(z) = \frac{z}{(1-z)^\beta} * f(z)$$

for $\beta \geq 0$. $(D^1 f)(z) = f(z)$, $(D^2 f)(z) = zf'(z)$ and for $\beta = n \in \mathbb{N}$ we have $D^{n+1} f = \frac{1}{n!} z(z^{n-1} f)^{(n)}$.

**Definition 1.15.2.** *Let $\alpha \leq 1$ and $\mathfrak{p} \in \mathbb{P}$ in $\mathbb{D}$ with $\mathfrak{p}'(0) > 0$, which $\mathfrak{p}(\mathbb{D})$ is starlike with respect to 1 and symmetric with respect to the real axis. Then the class $R^u_\alpha(\mathfrak{p})$ consists of all analytic function $f(z) \in \mathbf{H}(\Omega^*)$ satisfying*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \mathfrak{p}$$

Taking $g(z) = z + \sum_{n=2}^{\infty} n^m z^n$ for $m = 0, 1, 2, 3, \cdots$, then $f * g$ denotes the Sălăgean derivative of $f$ that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

**Lemma 1.15.1.** *(Ravichandran et al. [67]) If function $\mathfrak{p}(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{D}$ for which $\mathbf{Re}\,\mathfrak{p}(z) > 0$, then $|c_2 - \epsilon c_1^2| \leq 2 \max\{1, |2\epsilon - 1|\}$. The inequality is sharp for $p(z) = \dfrac{1+z}{1-z}$.*

**Remark 1.15.1.** *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \int_0^1 \frac{d\mu(t)}{1 - tz}$$

*where $a_n = \int_0^1 t^n d\mu(t)$, and $\mu(t)$ is a probability measure on $[0, 1]$.*

**Definition 1.15.3.** *Let $f$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of $f$ of order $\lambda$ is defined by*

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} \qquad (0 < \lambda < 1)$$

*where the multiplicity of $(z - \zeta)^\lambda$ is removed by requiring that $\log(z - \zeta)$ is real for $z - \zeta > 0$.*

Using the above definition and it's known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava introduced the operator $\Omega^\lambda := \mathbf{H}(\mathbb{D}) \to \mathbf{H}(\mathbb{D})$ for $\lambda$ any positive real number $\neq 2, 3, 4, \cdots$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2 - \lambda) z^\lambda D_z^\lambda f(z)$$

Here is defined the prestarlike Functions of Complex Order [77]

**Definition 1.15.4.** *Let $\alpha \leq 1$ and $b \neq 0$ be a complex number. Let $\mathfrak{p} \in \mathbb{P}$ in $\mathbb{D}$ with $\mathfrak{p}'(0) > 0$, which $\mathfrak{p}(\mathbb{D})$ is starlike with respect to 1 and symmetric with respect to the real axis. Then the class $R^{u}_{\alpha,b}(\mathfrak{p})$ consists of all analytic function $f(z) \in \mathbf{H}(\Omega^*)$ satisfying*

$$1 + \frac{1}{b}\Big(\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f} - 1\Big) \prec \mathfrak{p}$$

## 1.16 m-fold symmetric

A function $f(z)$ analytic in $\mathbb{D}$ is said to be m-fold symmetric $m = 2, 3, \cdots$, if $f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z)$. In particular, every odd $f(z)$ is 2-fold symmetric. Let $\mathcal{S}_m$ denote the subclass of $\mathcal{S}$ consisting of those $f(z)$ that are m-fold symmetric. It's clear that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots$$

## 1.17 $Q_\lambda(\beta)$

Ding et al. (1995) introduced the following class $Q_\lambda(\beta)$ of analytic functions [20]:

$$Q_\lambda(\beta) = \{f \in \mathcal{A} : \mathbf{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta\,,\ 0 \leq \beta < 1,\ \lambda \geq 0\} \tag{1.53}$$

We can see $Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$, Thus, for $\lambda \geq 1$, $0 \leq \beta < 1$ and $Q_\lambda(\beta) \subset Q_1(\beta) = \{f \in \mathcal{A} : \mathbf{Re}\, f'(z) > \beta,\ 0 \leq \beta < 1$ and hence $Q_\lambda(\beta)$ is univalent class.

## 1.18 averaging operators

Let $\mathbf{H}$ be the class of analytic functions in $\mathbb{D}$ and $co(E)$ denote the convex hull of a set $E$ in $\mathcal{C}$. Miller and Mocanu [50] introduced the concept of an *averaging operator defined on a set* $\mathbf{K} \subset \mathbf{H}$. The averaging operators is an operator $I : \mathbf{K} \to \mathbf{H}$ that satisfies $I[f](0) = f(0)$ and

$$I[f](\mathbb{D}) \subset co\big(f(\mathbb{D})\big) \tag{1.54}$$

for all $f \in \mathbf{K}$. A necessary and sufficient condition for averaging operator is

**Lemma 1.18.1.** *( [50], Lemma 2) Let $\mathbf{K} \subset \mathbf{H}$ and let an operator $I : \mathbf{K} \to \mathbf{H}$ satisfy $I[f](0) = f(0)$ for all $f \in \mathbf{K}$. A necessary and sufficient condition for $I$ to be an averaging operator on $\mathbf{K}$ is that*

$$(f \in \mathbf{K}, h\, convex, \text{ and } f \prec h) \Longleftrightarrow I[f] \prec h. \tag{1.55}$$

Also, the authors gave the example

$$I_\gamma[f](z) = \frac{\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \tag{1.56}$$

which is an averaging operator on $\mathbf{H}$ where $\mathbf{Re}\,\gamma > 0$. They gave [51] a generalization of (1.56) of the form

$$I_{\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}} \tag{1.57}$$

where $\mathbf{Re}\,\gamma > 0$ and showed that these operators are averaging operators on certain subsets of H we need to e

## 1.19 lowner chain

The following lemma concerns subordination (or Loewner) chains.

**Definition 1.19.1.** *([62], p.157) A function $L(z,t)$, $z \in \mathbb{D}$, $t \geq 0$ is a* subordination chain *if $L(0,t)$ is analytic and univalent in $\mathbb{D}$ for all $t \geq 0$, $L(z,0)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{D}$, and $L(z,s) \prec L(z,t)$ when $0 \leq s \leq t$.*

**Lemma 1.19.1.** *([62], p.159) The function $L(z,t) = a_1(t)z + \cdots$, with $z \in \mathbb{D}$ and $t \geq 0$, where $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$, is a subordination chain iff*

$$\mathbf{Re}\left[\frac{z\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right] > 0$$

*for all $z \in \mathbb{D}$ and $t \geq 0$.*

this special mapping from $\mathbb{D}$ onto a slit domain, plays a crucial role in both our lemma and the main theorem of this article.

Let $c \in \mathcal{C}$ with $\mathbf{Re}\, c > 0$ and let $N = N(c) = \dfrac{|c|(1+2\mathbf{Re}\, c)^{\frac{1}{2}} + \mathbf{Im} c}{\mathbf{Re}\, c}$. If $h$ is the univalent function $h(z) = \dfrac{2Nz}{1-z^2}$ and $b = h^{-1}(c)$ then let

$$Q_c(z) = h\Big(\frac{z+b}{1+\bar{b}z}\Big) = 2N\frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2 - (z+b)^2} \tag{1.58}$$

$z \in \mathbb{D}$. The function $Q_c$ is univalent in $\mathbb{D}$, $Q_c(0) = c$, and $Q_c(\mathbb{D}) = h(\mathbb{D})$ is the complex plane slit along the half-lines $\mathbf{Re}\, w = 0$, $\mathbf{Im} w \geq N$ and $\mathbf{Re}\, w = 0$, $\mathbf{Im} w \leq -N$.

# Chapter 2

# Bi-Univalent functions

Every univalent function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by $f^{-1}(f(z)) = z$ in $\mathbb{D}$. The Koebe one-quarter theorem 1.0.1 ensures that the image of $\mathbb{D}$ under every $f \in \mathcal{S}$ contains a disk of radius $\dfrac{1}{4}$, so we have $f(f^{-1}(w)) = w$ with $|w| < r_0(f)$ for which $r_0(f) \geq \frac{1}{4}$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (2.1)$$

**Definition 2.0.1.** *A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{D}$. We denote the class of all bi-univalent functions in $\mathbb{D}$ by $\Sigma$, where $f \in \mathcal{S}$ and $f^{-1}$ given by (2.1).*

$$\Sigma = \{f \in \mathcal{S} : f^{-1} \in \mathcal{S}\}. \quad (2.2)$$

Some examples of functions in the class $\Sigma$ are $\ell(z) = \dfrac{z}{1-z}$, $-\log(1-z)$ and $\dfrac{1}{2}\log\dfrac{1+z}{1-z}$ and some examples in class $\mathcal{S}$ that are not in the class $\Sigma$ are $k_0(z) = \dfrac{z}{(1-z)^2}$ and its rotations, $z - \dfrac{1}{2}z^2$ and $\dfrac{z}{1-z^2}$.

Lewin [44] in 1967 introduced and investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$ for all $f \in \Sigma$. Brannan and Clunie [D.A. Brannan, J.G. Clunie (Eds.), Aspects of Contemporary ... 1980.] (1980) conjectured that $|a_2| < \sqrt{2}$ and Netanyahu [54] showed that $\max\limits_{f\in\Sigma} |a_2| = \dfrac{4}{3}$. The coefficients bound estimate problem for each of the coefficients $a_n$ is still an open problem.

Brannan and Taha [13, 14] introduced certain subclasses of the bi-univalent function of class $\Sigma$, named strongly bi-starlike functions of order $\alpha$:

$$\mathcal{S}^*_\Sigma[\alpha] = \{f \in \Sigma : \left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2}\,, \left|\arg \frac{zg'(w)}{g(w)}\right| < \alpha\frac{\pi}{2}\,; 0 < \alpha \leq 1,\, z \in \mathbb{D}\} \tag{2.3}$$

We have also bi-starlike functions of order $\alpha$:

$$\mathcal{S}^*_\Sigma(\alpha) = \{f \in \Sigma : \mathbf{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\,, \mathbf{Re}\left(\frac{zg'(w)}{g(w)}\right) > \alpha\,; 0 \leq \alpha < 1,\, z \in \mathbb{D}\} \tag{2.4}$$

$\mathcal{H}_\Sigma(\beta)$ Srivastava et al. (2010) [83]

$$\mathbf{Re}\, f'(z) > \beta \quad ; \quad f(z) \in \Sigma, \quad z \in \mathbb{D} \tag{2.5}$$

$$\mathbf{Re}\, g'(w) > \beta \quad ; \quad g(w) = f^{-1}(w), \quad w \in \mathbb{D} \tag{2.6}$$

$0 \leq \beta < 1$. For $f \in \mathcal{H}_\Sigma(\beta)$:

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}} & , \; 0 \leq \beta < \frac{1}{3} \\ 1-\beta & , \; \frac{1}{3} \leq \beta < 1 \end{cases}, \; |a_3| \leq \frac{2(1-\beta)}{3} \tag{2.7}$$

$\mathcal{K}_\Sigma(\beta)$ Brannan and Taha (1986) [13]

$$\mathbf{Re}\, 1 + z\frac{f''(z)}{f'(z)} > \beta \quad ; \quad f(z) \in \Sigma, \quad z \in \mathbb{D} \tag{2.8}$$

$$\mathbf{Re}\, 1 + w\frac{g''(w)}{g'(w)} > \beta \quad ; \quad g(w) = f^{-1}(w), \quad w \in \mathbb{D} \tag{2.9}$$

$0 \leq \beta < 1$. For $f \in \mathcal{K}_\Sigma(\beta)$:

$$|a_2| \leq 1-\beta,\; |a_3| \leq \begin{cases} 1-\beta & , \; 0 \leq \beta < \frac{1}{3} \\ \frac{(1-\beta)(4-3\beta)}{3} & , \; \frac{1}{3} \leq \beta < 1 \end{cases} \tag{2.10}$$

$\mathcal{S}_\Sigma(\alpha, \beta)$ Aziz, Ebadian and Najafzadeh (2015) [9]

$$\mathbf{Re}\left((1-\alpha)f'(z) + \alpha(1 + z\frac{f''(z)}{f'(z)})\right) > \beta \quad ; \quad f(z) \in \Sigma, \quad z \in \mathbb{D} \tag{2.11}$$

$$\mathbf{Re}\left((1-\alpha)g'(w) + \alpha(1 + w\frac{g''(w)}{g'(w)})\right) > \beta \quad ; \quad g(w) = f^{-1}(w), \quad w \in \mathbb{D} \tag{2.12}$$

$0 \leq \alpha < 2$, $0 \leq \beta < 1$. For $f \in \mathcal{S}_\Sigma(\alpha,\beta)$:

$$|a_2| \leq \min\left\{1-\beta, \sqrt{\frac{2(1-\beta)}{3-\alpha}}\right\},\ |a_3| \leq \min\left\{\frac{2(1-\beta)}{3(1+\alpha)} + (1-\beta)^2, \frac{2(1-\beta)}{3-\alpha}\right\} \quad (2.13)$$

$\alpha = 0$: $\mathcal{S}_\Sigma(0,\beta) = \mathcal{H}_\Sigma(\beta)$ (Srivastava et al. [83])
$\alpha = 1$: $\mathcal{S}_\Sigma(0,\beta) = \mathcal{K}_\Sigma(\beta)$ (Brannan and Taha [13])

$\mathcal{N}^{\mu}_{\Sigma}(\alpha,\lambda)$ Branan, Taha (20??) [?]

$$\left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \quad ; \quad f(z) \in \Sigma,\quad z \in \mathbb{D} \quad (2.14)$$

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \quad ; \quad g(w) = f^{-1}(w),\quad w \in \mathbb{D} \quad (2.15)$$

$0 < \alpha \leq 1$, $\mu \geq 0$ and $\lambda \geq 1$. For $f \in \mathcal{N}^{\mu}_{\Sigma}(\alpha,\lambda)$:

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\mu+\lambda)^2 + \alpha(\mu+2\lambda-\lambda^2)}},\ |a_3| \leq \frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha}{2\lambda+\mu} \quad (2.16)$$

$\mu = 1$: $\mathcal{N}^{1}_{\Sigma}(\alpha,\lambda) = \mathcal{B}_\Sigma(\alpha,\lambda)$ (Frasin and Aouf 2011. [?])

$$\mathbf{Re}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right\} > \alpha\ ,\ \mathbf{Re}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right\} > \alpha$$

with $0 \leq \alpha < 1$. So

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1+2\lambda-\lambda^2)}},\ |a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1} \quad (2.17)$$

$\lambda = 1$: $\mathcal{N}^{\mu}_{\Sigma}(\alpha,1) = \mathcal{P}_\Sigma(\alpha,\mu)$ (Prima and Keresi 2013. [?])
is Bi-Bazelivić functions, satisfy condition $f \in \mathcal{P}_\Sigma(\alpha,\mu)$:

$$\mathbf{Re}\,\frac{z^{1-\mu}f'(z)}{f(z)^{1-\mu}} > \alpha\ ,\ \mathbf{Re}\,\frac{w^{1-\mu}g'(w)}{g(w)^{1-\mu}} > \alpha$$

with $0 \leq \alpha < 1$. So

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\mu+1)^2 + \alpha(\mu+1)}},\ |a_3| \leq \frac{4\alpha^2}{(\mu+1)^2} + \frac{2\alpha}{\mu+2} \quad (2.18)$$

$\lambda = \mu = 1$: $\mathcal{N}_{\Sigma}^{1}(\alpha, 1) = \mathcal{H}_{\Sigma}(\alpha)$ (Srivastava et al. 2010. [83])

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}}, \ |a_3| \leq \frac{1}{3}\alpha(3\alpha+2) \tag{2.19}$$

$\lambda = 1$ and $\mu = 0$: $\mathcal{N}_{\Sigma}^{0}(\alpha, 1) = \mathcal{S}_{\Sigma}^{*}(\alpha)$ (Bi-Stralike function of order $\alpha$)

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}}, \ |a_3| \leq \alpha(4\alpha+1) \tag{2.20}$$

Bulut (2014) shows with Faber coefficients for $n \geq 4$:

$$|a_n| \leq \frac{2(1-\alpha)}{\mu+(n-1)\lambda}$$

and

$$|a_3 - \frac{\mu+3}{2}a_2^2| = \frac{2(1-\alpha)}{\mu+2\lambda}$$

Bulut also obtain beter bounds for $a_2$ and $a_3$ for function in these classes (2014). Zhu (2007) determine conditions on $\alpha$, $\lambda$, $\mu$ and $M$ which

$$\Big|(1-\lambda)\big(\frac{f(z)}{z}\big)^{\mu} + \lambda f'(z)\big(\frac{f(z)}{z}\big)^{\mu-1} - 1\Big| < M$$

$\mathcal{K}_{\Sigma}(\alpha, \beta)$ Sim and Kwon (**2013**) [**81**]

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$$\alpha < \mathbf{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < \beta \quad ; \quad f(z) \in \Sigma, \quad z \in \mathbb{D} \tag{2.21}$$

$$\alpha < \mathbf{Re}\left(1 + z\frac{g''(z)}{g'(z)}\right) < \beta \quad ; \quad g(w) = f^{-1}(w), \quad w \in \mathbb{D} \tag{2.22}$$

For real numbers $\alpha$ and $\beta$ with $0 \leq \alpha < 1 < \beta$, For $f \in \mathcal{K}_{\Sigma}(\alpha, \beta)$:

$$|a_2| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{2|B_1^2 - 2B_1 + 2B_2|}}, \ |a_3| \leq \frac{1}{2}(|B_1| + |B_2 - B_1|) \tag{2.23}$$

where $|B_1| = i\dfrac{\beta-\alpha}{\pi}\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)$, $|B_2| = i\dfrac{\beta-\alpha}{2\pi}\left(1 - e^{4\pi i\frac{1-\alpha}{\beta-\alpha}}\right)$ and

$$|b_2| \leq \frac{\beta-\alpha}{\pi}\sin\frac{(1-\alpha)\pi}{\beta-\alpha}, \tag{2.24}$$

$$|b_3| \leq \frac{\beta-\alpha}{3\pi}\sin\frac{(1-\alpha)\pi}{\beta-\alpha}\max\left\{1, \left|\frac{1}{2} - \frac{2i(\beta-\alpha)}{\pi} + (\frac{1}{2} + \frac{2i(\beta-\alpha)}{\pi})e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right|\right\} \tag{2.25}$$

# Chapter 3

# Univalent harmonic functions

Harmonic functions have been studied by differential geometers such as Choquet, Kneser, Lewy, and Rado in past century. Then harmonic complex functions investigated by geometric function theorists Clunie and Sheil-Small [18]. They developed the basic theory of the family of harmonic functions which are univalent in $\mathbb{D}$. Later studies showed further properties in this family of complex-valued functions. In following we will note most of properties harmonic univalent maps.

## 3.1 Real-Value Harmonic Univalent Maps

In this context, a function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is called real harmonic if $u_{xx} + u_{yy} = 0$, that is Laplace equation. The Laplace equation arises in many applications, for example in physics in the following scenarios. In hydrodynamics, the "velocity potential" of the fluid flow satisfies the Laplace equation, while in electrostatics, the electrostatic potential satisfies the Laplace equation. The Laplace equation also has an important link with stochastic processes.

A harmonic function is $\mathbb{C}^\infty$, they are infinitely many times differentiable, and every harmonic function in a simply connected domain is the real part of some holomorphic function defined there. Furthermore

**Lemma 3.1.1.** *[30] Harmonic functions have the Mean Value Property, and hence they satisfy the Maximum Modulus Principle. Real-valued harmonic functions also satisfy the Maximum and Minimum Principles.*

**Lemma 3.1.2.** *(***The Poisson integral formula** *[30]). Let $f$ be a harmonic function on the domain $|z| < \rho$ for some $\rho > 0$. Then, for each $0 < r < \rho$,*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} f(re^{i\theta})d\theta, \qquad for\ |z| < r. \tag{3.1}$$

## 3.2 Complex-Value Harmonic Univalent Maps

In folloing we will discuss about planar harmonic mappings. These functions can be thought of as a generalization of analytic maps.

We let $H$ denote the family of continuous complex-valued functions which are harmonic in the open unit disk $\mathbb{D}$.

Let $f : D \subset \mathbb{C} \to \mathbb{C}$. The function $f(x, y) = u(x, y) + iv(x, y)$ is *complex-valued harmonic function* if $f$ is continuous and $u$ and $v$ are real harmonic in $D$. Clearly, $u$ and $v$ aren't harmonic conjugates otherwise $f$ will be a analyic function and isn't our cast. If $D$ be a simply-connected, so we have a useful representation for $f$ [18]:

**Lemma 3.2.1.** *Let $D$ be a simply-connected domain and $f = u + iv$ is harmonic in $D$, then $f$ has a canonical representation $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$, the co-analytic part of $f$.*

In $\mathbb{D}$, $g$ and $h$ can be expanded in Taylor series as $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ so we may represent $f$ by a power series of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{3.2}$$

The *Jacobian* of a function $f = u + iv$ is $J_f(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x =$

$|f_z|^2 - |f_{\overline{z}}|^2 = |h'(z)|^2 - |g'(z)|^2$, since

$$f_z = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) = \frac{1}{2}(u_x + iv_x - iu_y + v_y) = \frac{1}{2}\Big(u_x + v_y + i(v_x - u_y)\Big)$$
$$f_{\overline{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big) = \frac{1}{2}(u_x + iv_x + iu_y - v_y) = \frac{1}{2}\Big(u_x - v_y + i(v_x + u_y)\Big)$$
$$|f_z|^2 - |f_{\overline{z}}|^2 = \frac{1}{4}\Big((u_x + v_y)^2 + (v_x - u_y)^2 - (u_x - v_y)^2 - (v_x + u_y)^2\Big) = u_x v_y - u_y v_x$$

If f is analytic, its Jacobian takes the form $J_f(z) = u_x^2 + v_x^2 = |f'(z)|^2$. A harmonic map $f = h + \overline{g}$ is s sense-preserving at $z = a$ if $J_f(a) > 0$, that is $|h'(a)|^2 - |g'(a)|^2 > 0$ or $|h'(a)| > |g'(a)|$. Let $\omega(z) = \dfrac{g'(z)}{h'(z)}$, so $f = h + \overline{g}$ is a sense-preserving at $z = a$ if $|\omega(a)| < 1$ and it is sense-reserving if $|\omega(a)| > 1$. The function $\omega(z)$ is an analytic function, called (second complex) *dilatation* of $f$. Note that $\omega(z) = 0$ if and only if f is analytic. We see that $f = h + \overline{g}$ is equivalent to $f = \mathbf{Re}\,(h + g) + i\mathbf{Im}(h - g)$.

**Lemma 3.2.2.** *[23] All critical points of a nonconstant harmonic function are isolated.*

**Theorem 3.2.1.** *(Lewy's Theorem [23]) If $f$ is a complex-valued harmonic function that is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian $J_f(z) \neq 0$ for all $z \in D$.*

So, in the Lewy's theorem view, for a complex-valued harmonic function to be locally univalent and sense-preserving in a domain, $J_f > 0$ in domain. For $J_f < 0$, $\overline{f}$ is sense-preserving.

The Poisson integral formula also is true for complex-value harmonic Maps [23].

## 3.3 $\mathcal{S}_H$ Class

Let $\mathcal{S}_H$ denote the class of complex-valued univalent harmonic functions of form (3.2) that are sense-preserving on $\mathbb{D}$, and normalized by $a_0 = 0$, $a_1 = 1$. So we can write every $f \in \mathcal{S}_H$ in the form of

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\,\overline{z}^n \tag{3.3}$$

## 3.4 $\mathcal{S}_H^0$ Class

Furthermore if we restrict $f \in \mathcal{S}_H$ to have $b_1 = 0$, this subclass is shown by $\mathcal{S}_H^0$. Clearly $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$. $\mathcal{S}_H^0$ is a compact normal family, but $\mathcal{S}_H$ not so.

$f_{\overline{z}}(0) = \overline{g'(0)} = \overline{b_1} = 0$

This class is preserved under conjugation, rotation, dilation, disc automorphism and range transformation!

Like as Bieberbach conjecture in the class $\mathcal{S}$, we have *harmonic Bieberbach conjecture*: Let $f(z) \in \mathcal{S}_H^0$ be a harmonic function of form (3.3) then

$$|a_n| \leq \frac{1}{6}(n+1)(2n+1) \tag{3.4}$$

$$|b_n| \leq \frac{1}{6}(n-1)(2n-1) \tag{3.5}$$

$$\big||a_n| - |b_n|\big| \leq 1 \tag{3.6}$$

However this Conjecture is an open problem, the best known bound for all functions $f(z) \in \mathcal{S}_H^0$ is $|a_2| < 49$ and also

**Lemma 3.4.1.** *[23] For all functions $f(z) \in \mathcal{S}_H^0$, the sharp inequality $|b_2| \leq \frac{1}{2}$ holds.*

## 3.5 Shearing Technique

Shear construction devised by Clunie and Sheil-Small [18] in 1984 for more study in planar harmonc functions. This technique provides a procedure for constructing univalent harmonic maps $f = h + \overline{g} \in \mathcal{S}_H$. The shear technique is essential to the work on harmonic mapps, because it allows us to study harmonic functions by examining their related analytic functions.

A domain $D \subset \mathbb{C}$ is said to be *convex in the horizontal direction* (CHD) if its intersection with each horizontal line is connected(or empty), so

**Lemma 3.5.1.** *[23] Let $f = h + \overline{g}$ be harmonic and locally univalent in $\mathbb{D}$. Then $f$ is univalent and its range is CHD if and only if $h - g$ be univalent and its range be CHD.*

For use of shearing technique we use above lemma and with univalent $f = h + \overline{g}$ and CHD range, make a univalent $F = h - g$ and CHD range to have dilatation $\omega(z) = \dfrac{g'(z)}{h'(z)}$. In this process the assumption $|\omega(z)| < 1$ is necessary to be $f$ sense-preserving (Theorem 3.2.1).

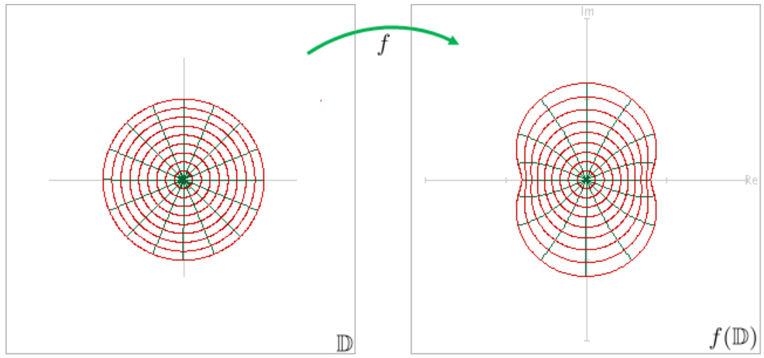



Figure 3.1: The image of $\mathbb{D}$ under $F = z - \frac{1}{6}z^3$ map.

**Example 3.5.1.** *Let* $F = z - \dfrac{1}{6}z^3$ *that is univalent with CHD range (Fig.3.1). We want to construct univalent harmonic map* $f = h + \overline{g}$ *with dilatation* $\omega(z) = \dfrac{1}{\sqrt{2}}z$*. Then* $f$ *will be univalent harmonic maps and sense-preserving* $|\omega(z)| = |\dfrac{1}{\sqrt{2}}z| < 1$*. Write* $h - g = z - \dfrac{1}{6}z^3$ *and* $\dfrac{g'(z)}{h'(z)} = \dfrac{1}{\sqrt{2}}z$ *so,*

$$\begin{cases} h' - g' = 1 - \frac{1}{2}z^2 \\ g' = \frac{1}{\sqrt{2}}zh' \end{cases} \quad \begin{cases} h'(z) = 1 + \frac{1}{\sqrt{2}}z \\ g'(z) = \frac{1}{\sqrt{2}}z + \frac{1}{2}z^2 \end{cases} \quad \begin{cases} h(z) = z + \frac{1}{2\sqrt{2}}z^2 \\ g(z) = \frac{1}{2\sqrt{2}}z^2 + \frac{1}{6}z^3 \end{cases}$$

*note that* $h(0) = g(0) = 0$ *and then* $f = h + \overline{g} = z + \dfrac{1}{2\sqrt{2}}z^2 + \dfrac{1}{2\sqrt{2}}\overline{z}^2 + \dfrac{1}{6}\overline{z}^3$ *is desired harmonic map that is univalent and CHD (Fig.3.2).*

**Example 3.5.2.** *Let* $F = \dfrac{z}{1-z}$ *that is convex univalent (Fig.1.2). We construct univalent harmonic map* $f = h + \overline{g}$ *with dilatation* $\omega(z) = z^2$*. From* $|\omega(z)| = |z^2| < 1$ *ensure local univalence of* $f$*. Get* $h - g = \dfrac{z}{1-z}$ *and* $\dfrac{g'(z)}{h'(z)} = z^2$ *and then,*

$$\begin{cases} h' - g' = \frac{1}{(1-z)^2} \\ g' = z^2 h' \end{cases} \quad \begin{cases} h'(z) = \frac{1}{(1-z^2)(1-z)^2} \\ g'(z) = \frac{z^2}{(1-z^2)(1-z)^2} \end{cases} \quad \begin{cases} h(z) = \frac{z-2}{4(z-1)^2} + \frac{1}{8}\log\frac{z+1}{z-1} + \frac{i\pi}{8} - \frac{1}{2} \\ g(z) = \frac{3z-2}{4(z-1)^2} + \frac{1}{8}\log\frac{z+1}{z-1} + \frac{i\pi}{8} + \frac{1}{2} \end{cases}$$

*with integration constant* $h(0) = g(0) = 0$ *and then* $f = h + \overline{g} = -\dfrac{1}{2}\dfrac{z}{(1-z)^2} +$

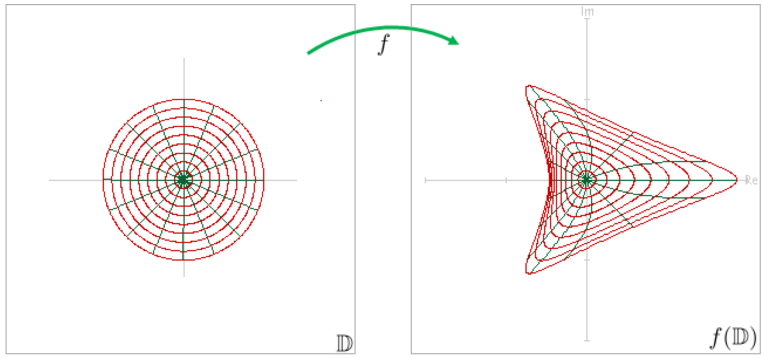



Figure 3.2: The image of $\mathbb{D}$ under $f = z + \frac{1}{2\sqrt{2}}z^2 + \frac{1}{2\sqrt{2}}\overline{z}^2 + \frac{1}{6}\overline{z}^3$ map.

$2\mathbf{Re}\left(\dfrac{3z-2}{4(z-1)^2}+\dfrac{1}{8}\log\dfrac{z+1}{z-1}\right)$ *is desired CHD univalent harmonic map (Fig.3.3).*

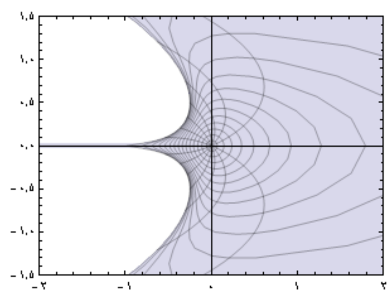
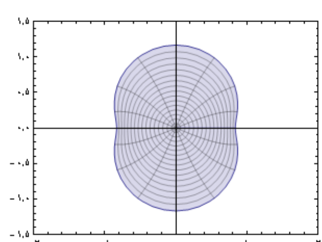
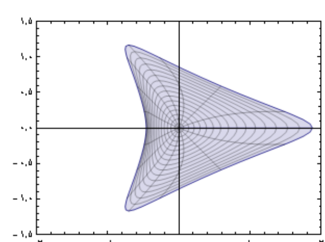
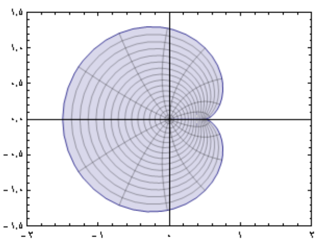
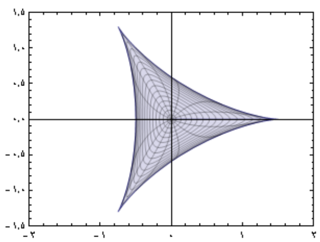
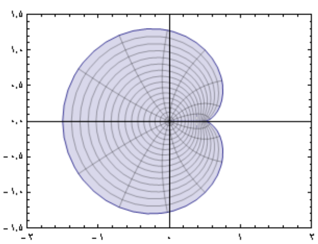
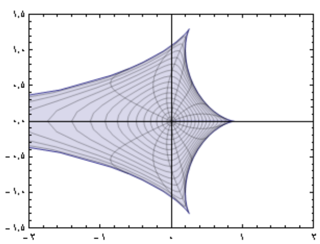
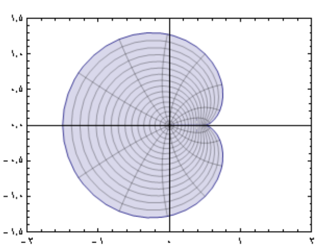
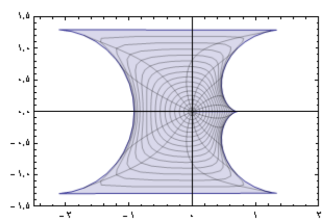
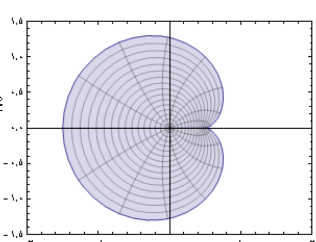
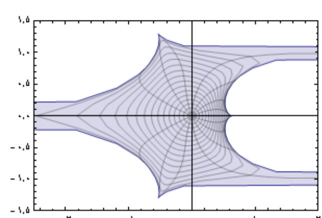
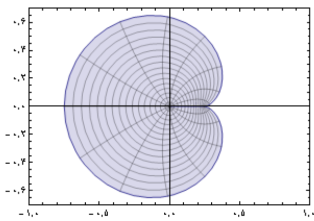
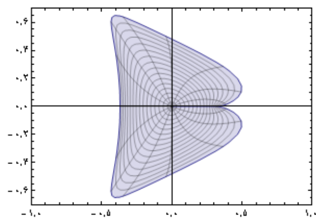
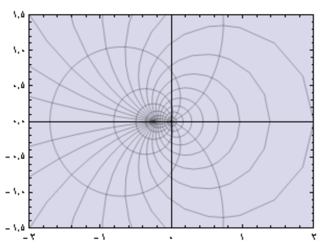
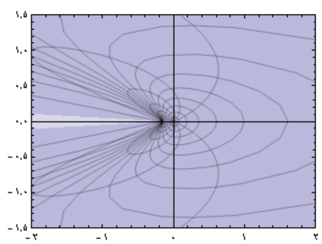


Figure 3.3: The image of $\mathbb{D}$ under harmonic map $f$ in example 3.5.2.

We have some examples in the following table for some analytic CHD maps $F$ with given dilatation $\omega$ and obtained corresponding harmonic functions $f$. In every case have drawn the graphs of $F$ and $f$.

| Technique | $F$ | $f$ |
|---|---|---|
| $\begin{cases} F = z - \frac{1}{6}z^3 \\ \omega = \frac{1}{\sqrt{2}}z \\ h = z + \frac{1}{2\sqrt{2}}z^2 \\ g = \frac{1}{2\sqrt{2}}z^2 + \frac{1}{6}z^3 \\ f = z + \frac{1}{2\sqrt{2}}z^2 \\ +\frac{1}{2\sqrt{2}}\overline{z}^2 + \frac{1}{6}\overline{z}^3 \end{cases}$ |  |  |

| | | |
|---|---|---|
| $\begin{cases} F = z - \frac{1}{2}z^2 \\ \omega = z \\ h = z \\ g = \frac{1}{2}z^2 \\ f = z + \frac{1}{2}\overline{z}^2 \end{cases}$ |  |  |
| $\begin{cases} F = z - \frac{1}{2}z^2 \\ \omega = z^2 \\ h = \log(1+z) \\ g = -z + \frac{1}{2}z^2 \\ \quad + \log(1+z) \\ f = 2\mathbf{Re}\, \log(1+z) \\ \quad - \overline{z} + \frac{1}{2}\overline{z}^2 \end{cases}$ |  |  |
| $\begin{cases} F = z - \frac{1}{2}z^2 \\ \omega = z^3 \\ h = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2z+1}{\sqrt{3}}) \\ \quad - \frac{\pi}{3\sqrt{3}} \\ g = -z + \frac{1}{2}z^2 - \frac{\pi}{3\sqrt{3}} \\ \quad + \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2z+1}{\sqrt{3}}) \\ f = h + \overline{g} \end{cases}$ |  |  |
| $\begin{cases} F = z - \frac{1}{2}z^2 \\ \omega = z^4 \\ h = \frac{1}{4}\log\frac{(1+z)^2}{1+z^2} + \frac{\tan^{-1} z}{2} \\ g = -z + \frac{1}{2}z^2 \\ +\frac{1}{4}\log\frac{(1+z)^2}{1+z^2} + \frac{\tan^{-1} z}{2} \\ f = h + \overline{g} \end{cases}$ |  |  |

| | | |
|---|---|---|
| $\begin{cases} F = z - \frac{1}{2}z^2 \\ \omega = \frac{1}{2}z - \frac{1}{4}z^2 \\ h = \log \frac{4}{z^2-2z+4} \\ g = -\frac{1}{2}z + \frac{1}{4}z^2 \\ \quad + \log \frac{4}{z^2-2z+4} \\ f = h + \overline{g} \end{cases}$ |  |  |
| $\begin{cases} k_0 = \frac{z}{(1-z)^2} \\ \omega = z \\ h = \frac{z-\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3} \\ g = \frac{\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3} \\ f = \mathbf{Re}\frac{z+\frac{1}{3}z^3}{(1-z)^3} \\ \quad + i\mathbf{Im}\frac{z}{(1-z)^2} \end{cases}$ |  |  |

In the last row, we have got the Koebe function $k_0(z) = \frac{z}{(1-z)^2}$ and make univalent harmonic map $f(z) = \mathbf{Re}\,\frac{z+\frac{1}{3}z^3}{(1-z)^3} + i\mathbf{Im}\frac{z}{(1-z)^2}$ with dilatation $\omega(z) = z$. $f$ is known as *harmonic Koebe function*, that is

$$k_{0H}(z) = \mathbf{Re}\,\frac{z+\frac{1}{3}z^3}{(1-z)^3} + i\mathbf{Im}\frac{z}{(1-z)^2} = \frac{z-\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3} + \frac{\frac{1}{2}\overline{z}^2+\frac{1}{6}\overline{z}^3}{(1-\overline{z})^3} \in \mathcal{S}_H^0 \quad (3.7)$$

The harmonic Koebe function maps $\mathbb{D}$ onto the plane minus the negative real half-line from $-\dfrac{1}{6}$ to $\infty$ as has shown in it's graph [23]. the coefficients of $h$ and $g$ are found to be

$$|a_n| \leq \frac{1}{6}(n+1)(2n+1) \quad , \quad |b_n| \leq \frac{1}{6}(n-1)(2n-1) \quad (3.8)$$

So, this conjecture that for all functions $f \in \mathcal{S}_H^0$ and for all indices $n$, the coefficients of $h$ and $g$ must to satisfy in (3.8) is an open problem. It suggests that perhaps each function in $\mathcal{S}_H^0$ will cover the disk $|w| < \dfrac{1}{6}$, but,

**Lemma 3.5.2.** *[18] The range of every $f \in \mathcal{S}_H^0$ contains the disk $|w| < \dfrac{1}{16}$.*

Open Problem 4. Construct examples of harmonic univalent functions whose dilatation is a singular inner function and determine properties of these functions.

## 3.6 Starlikeness

A harmonic function $f(z)$ is said to be starlike if it's range be starlike with respect to the origin. This means that every point of the range can be connected to the origin by a radial line that lies entirely in the region. In the other words, $\arg\{f(e^{i\theta})\}$ will be a nondecreasing function of $\theta$, or that

$$\frac{\partial}{\partial\theta}\arg\{f(e^{i\theta})\} \geq 0$$

The class of all starlike functions in $\mathcal{S}_H$ is shown by $\mathcal{S}_H^*$. The subclass of all starlike functions in $\mathcal{S}_H^0$ denote by $\mathcal{S}_H^{0*}$.

**Lemma 3.6.1.** *(Silverman [79]) If* $\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1$, *then* $f \in \mathcal{S}_H^*$.

The subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ includes all functions $f \in \mathcal{S}_H$ with $f_{\overline{z}}(0) = 1$, so $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$. Clunie and Sheil-Small also considered starlike functions in $S_H$, denote by $\mathcal{S}_H^*$. The subclass of all starlike functions in $\mathcal{S}_H^0$ denote by $\mathcal{S}_H^{0*}$. Starlikeness isn't a hereditary property for harmonic mappings, so the image of every subdisk $|z| < r < 1$ is not necessarily starlike with respect to the origin [23, 3]. Thus we need a property to explain starlikeness of a map in a hereditary form. We have folowing definition.

**Definition 3.6.1.** *A harmonic mapping $f$ with $f(0) = 0$ is said to be fully-starlike if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin.*

For $f \in \mathcal{S}_H$, the family of fully-starlike functions denotes by $\mathcal{FS}_H^*$. In 1980 Mocanu gave a relation between fully-starlikeness and a differential operator of a non-analytic function [52]. Let

$$Df = zf_z - \overline{z}f_{\overline{z}} \tag{3.9}$$

be the differential operator and

$$D^2 f = D(Df) = zf_z + \overline{z}f_{\overline{z}} + zzf_{zz} + \overline{zz}f_{\overline{zz}} \tag{3.10}$$

**Lemma 3.6.2.** *Let $f \in C^1(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and* $\mathbf{Re}\,\frac{Df(z)}{f(z)} > 0$ *then $f$ is univalent and fully-starlike in $\mathbb{D}$.*

A harmonic mapping $f$ with $f(0) = 0$ is said to be fully starlike if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin. However, a fully starlike mapping need not be univalent [17].

Let $Df = zf_z - \overline{z}f_{\overline{z}}$ denote the differential operator and

$$D^2 f = D(Df) = zf_z + \overline{z}f_{\overline{z}} + zzf_{zz} + \overline{zz}f_{\overline{zz}}$$

For sense-preserving complex-valued function $f(z)$, $Df \neq 0$. If $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $f$ satisfies condition such as $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ or $\mathbf{Re}\,\dfrac{D^2f(z)}{Df(z)} > 0$ for all $z \in \mathbb{D} - \{0\}$, then $f$ maps every circle $0 < |z| = r < 1$ onto a simple closed curve [52], from

$$\frac{\partial}{\partial\theta}\arg\{f(e^{i\theta})\} = \mathbf{Re}\,\frac{Df(z)}{f(z)}$$

we have

**Lemma 3.6.3.** *(Mocanu [52]) Let $f \in C^1(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ then $f$ is univalent and fully starlike in $\mathbb{D}$.*

We denote by $\mathcal{S}_H^{0*}(\alpha)$ the subclass of $\mathcal{S}_H^0$ consisting of starlike functions of order $\alpha$ $(0 \leq \alpha < 1)$. Jahangiri [38] proved that $f \in \mathcal{S}_H^{0*}(\alpha)$ if

$$\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=2}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n| \leq 1 \tag{3.11}$$

## 3.7 Convexity

A harmonic function $f(z) \in \mathcal{S}_H$ is said to be convex if it's range is a convex set on $\mathbb{C}$. In geometric view of the range of $f(\mathbb{D})$, this means that $\arg\{\frac{\partial}{\partial\theta}f(e^{i\theta})\}$ be a nondecreasing function of $\theta$, or that

$$\frac{\partial}{\partial\theta}\arg\left\{\frac{\partial}{\partial\theta}f(e^{i\theta})\right\} \geq 0$$

The hereditary property for conformal maps does not generalize to harmonic mappings. If f is a univalent harmonic map of $\mathbb{D}$ onto a convex domain, then the image of the disk $|z| < r$ is convex for each radius $r \leq \sqrt{2} - 1$, but not necessarily for any radius in the interval $\sqrt{2}-1 < r < 1$. In fact, the function

$$\begin{aligned} f(z) &= \mathbf{Re}\frac{z}{1-z} + i\mathbf{Im}\frac{z}{(1-z)^2} \\ &= \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \frac{-\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2} \in \mathcal{K}_H \end{aligned} \tag{3.12}$$

is a harmonic mapping of the disk onto the half-plane $\mathbf{Re}\, w > -\frac{1}{2}$, but the image of the disk $|z| \leq r$ fails to be convex for every $r$ in the interval $\sqrt{2} - 1 < r < 1$ (Fig.3.4). The class of all convex maps in $\mathcal{S}_H$ is shown by

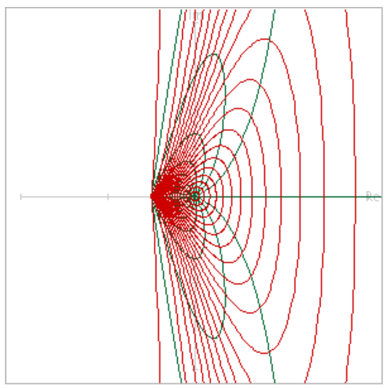


Figure 3.4: The image of $\mathbb{D}$ under harmonic map (3.12).

$\mathcal{K}_H$ and so on, the subclass $\mathcal{K}_H^0$ for $\mathcal{S}_H^0$.

**Lemma 3.7.1.** *(Clunie & Sheil-Small [18]) For $f$ given by (3.2), If $f \in \mathcal{K}_H$, then for $n \in \mathbb{N}$:*

$$|A_n| \leq \frac{n-1}{2}|B_1| + \frac{n+1}{2} \quad , \quad |B_n| \leq \frac{n-1}{2} + \frac{n+1}{2}|B_1|$$

*for $n \geq 2$: $|A_n| < n$, $|B_n| < n$.*

**Lemma 3.7.2.** *(Silverman [79]) If $\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq 1$, then $f \in \mathcal{K}_H$.*

A harmonic function $f$ is univalent and convex if and only if for each $\alpha \in \mathbb{R}$, the analytic function $e^{i\alpha}h - e^{i\alpha}g$ is univalent and convex in the horizontal direction.

A harmonic mapping of the unit disk will be called fully convex if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve. In particular, $f(z) \neq 0$ for $0 < |z| < 1$, according to the Rado-Kneser-Choquet theorem, a fully convex harmonic mapping is necessarily univalent in $\mathbb{D}$ [17]. If $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$, we have [52]

$$\frac{\partial}{\partial\theta} \arg\left\{\frac{\partial}{\partial\theta} f(e^{i\theta})\right\} = \mathbf{Re}\,\frac{D^2 f(z)}{Df(z)}$$

so,

**Lemma 3.7.3.** *(Mocanu [52]) Let $f \in C^2(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $\mathbf{Re}\,\dfrac{D^2 f(z)}{Df(z)} > 0$ then then $f$ is univalent and fully convex in $\mathbb{D}$.*

A univalent harmonic function $f$ in $D$ is said to be convex in the direction of $\alpha$ if $f(\mathbb{D})$ is convex in the direction of $\alpha$. We say that $f$ is *convex in one direction* if there exists an $\alpha$ such that $f$ is convex in the direction of $\alpha$.

We denote by $\mathcal{K}_H^0(\alpha)$ the subclass of $\mathcal{S}_H^0$ consisting of convex functions of order $\alpha$ $(0 \leq \alpha < 1)$.

**Lemma 3.7.4.** *(Jahangiri [38]) $f \in \mathcal{K}_H^0(\alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1 \tag{3.13}$$

**Lemma 3.7.5.** *(Jahangiri [38]) $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1$ then $f \in \mathcal{K}_H^0(\alpha)$.*

## 3.8 Close-to-Convex

A harmonic function $f(z) \in \mathcal{S}_H$ is said to be close-to-convex if its range $f(\mathbb{D})$ is a close-to-convex domain.

The class of all close-to-convex harmonic functions in $\mathcal{S}_H$ is shown by $\mathcal{C}_H$. The subclass of all close-to-convex functions in $\mathcal{S}_H^0$ denote by $\mathcal{C}_H^0$.

**Lemma 3.8.1.** *[18] Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + \epsilon g$ is convex for some $|\epsilon| \leq 1$. Then $f$ is univalent and close-to-convex in $\mathbb{D}$.*

## 3.9 Convolution

The convolution of two harmonic functions $f(z)$ and $F(z)$ with canonical representations

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n \tag{3.14}$$

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n \tag{3.15}$$

defined as

$$(f * F)(z) = (h * H)(z) + \overline{g * G(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n \tag{3.16}$$

Unlike to the case of analytic functions, the convolution of two harmonic functions $f(z)$ and $F(z)$, need not preserves the properties of this class. Let $f(z) = h(z) + \overline{g(z)} = \dfrac{z}{1-z}$ with dilatation $\omega(z) = -z$, so $f \in \mathcal{K}_H^0$ and $F(z) = H(z) + \overline{G(z)} = \dfrac{z}{1-z}$ with dilatation $\omega(z) = -z^n$, for $n \in \mathbb{N}$, so $F \in \mathcal{K}_H^0$ also, then $(f * F)(z)$ isn't locally univalent in $\mathbb{D}$, for $n \geq 3$ [22].

**Lemma 3.9.1.** *(Clunie and Shiel-Small [18]) If $\phi \in \mathcal{K}$ and $F \in \mathcal{K}_H$ then,*

$$(\phi + \epsilon\overline{\phi}) * F \in \mathcal{C}_H$$

*for $|\epsilon| \leq 1$.*

Ahuja et al. [2] showed that the required convexity condition for $\phi$ cannot be replaced by starlikeness. For example consider the starlike analytic function $\phi(z) = z + \dfrac{1}{n}z^n$ in $\mathbb{D}$ and $\epsilon = 0$. Let

$$F(z) = h + \overline{g} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \frac{-\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2} \in \mathcal{K}_H$$

Then the convolution function

$$(\phi + 0\overline{\phi}) * F = z + \frac{n+1}{2n}z^n \quad , \quad n \geq 2$$

which is not even univalent in $\mathbb{D}$.

**Lemma 3.9.2.** *(Ahuja et al. [2]) Let $h$ and $g$ be analytic in $\mathbb{D}$ so that $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex in $\mathbb{D}$ for each $|\epsilon| = 1$. Then*

$$h + \overline{g} \in \mathcal{C}_H$$

*If $\phi$ is convex analytic in $\mathbb{D}$, then*

$$(\phi + \sigma\overline{\phi}) * (h + \overline{g}) \in \mathcal{C}_H \quad , \quad |\sigma| = 1$$

Clunie and Shiel-Small posed the question for what harmonic functions $\phi$ is $\phi * f \in \mathcal{K}_H$, where $f \in \mathcal{K}_H$? This question was partially answered by Ruscheweyh and Salinas [74]. They proved that if $\phi$ is analytic in $\mathbb{D}$, then for all $F \in \mathcal{K}_H$, $\phi * F = \phi * \mathbf{Re}\, F + \overline{\phi * \mathbf{Im} F} \in \mathcal{K}_H$ iff for each real number $\gamma$, the function all $\phi + i\gamma z\phi'$ is convex in the direction of imaginary axis.

**Lemma 3.9.3.** *(Ahuja et al. [2]) Let $h$ and $\phi$ be convex analytic in $\mathbb{D}$, and $g$ is analytic there with $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Then for each $|\epsilon| \le 1$,*

$$(\phi + \epsilon\overline{\phi}) * (h + \overline{g}) \in \mathcal{C}_H$$

Following theorem gives necessary and sufficient convolution conditions for starlikeness of harmonic functions.

**Theorem 3.9.1.** *(Ahuja et al. [2]) Let $f = h + \overline{g} \in \mathcal{S}_H$, then $f \in \mathcal{S}_H^*$ iff*

$$h(z) * \frac{z + \frac{1}{2}(\zeta - 1)z^2}{(1-z)^2} - \overline{g(z)} * \frac{\zeta\overline{z} - \frac{1}{2}(\zeta - 1)\overline{z}^2}{(1-\overline{z})^2} \neq 0 \quad ; \quad |\zeta| = 1,\ 0 < |z| < 1$$

**Corollary 3.9.2.** *(Ahuja et al. [2]) Let $f = h + \overline{g} \in \mathcal{S}_H$, if $\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \le 1$ then $f \in \mathcal{S}_H^*$.*

Next theorem gives necessary and sufficient convolution conditions for convexness for harmonic functions.

**Theorem 3.9.3.** *(Ahuja et al. [2]) Let $f = h + \overline{g} \in \mathcal{S}_H$, then $f \in \mathcal{K}_H$ iff*

$$h(z) * \frac{z + \zeta z^2}{(1-z)^3} + \overline{g(z)} * \frac{\zeta\overline{z} + \overline{z}^2}{(1-\overline{z})^3} \neq 0 \quad ; \quad |\zeta| = 1,\ 0 < |z| < 1$$

**Corollary 3.9.4.** *(Ahuja et al. [2]) Let $f = h + \overline{g} \in \mathcal{S}_H$, if $\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq 1$ then $f \in \mathcal{K}_H$.*

Kumar et al. [42] have given some result based on coefficients inequalities for convex harmonic 3.7.1 that makes benefit conclusions about convolution:

**Lemma 3.9.4.** *[42] For three representation of $f$, $F$ and $f * F$ given by 3.14-3.16 we have*

- *If $\displaystyle\sum_{n=2}^{\infty} n^3|a_n| + \sum_{n=2}^{\infty} n^3|b_n| \leq 1$ and $F \in \mathcal{K}_H^0$, then $f * F \in \mathcal{K}_H^0$.*
- *[42] If $\displaystyle\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=2}^{\infty} n^2|b_n| \leq 1$ and $F \in \mathcal{K}_H^0$, then $f * F \in \mathcal{S}_H^{*0}$.*
- *[42] If $\displaystyle\sum_{n=2}^{\infty} n^3|a_n| + \sum_{n=2}^{\infty} n^3|b_n| \leq 1 - |b_1|$ and $F \in \mathcal{K}_H$, then $f * F \in \mathcal{K}_H$.*
- *[42] If $\displaystyle\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=2}^{\infty} n^2|b_n| \leq 1 - |b_1|$ and $F \in \mathcal{K}_H$, then $f * F \in \mathcal{S}_H^*$.*
- *[42] If $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1$ and $F \in \mathcal{K}_H^0$, then $f * F \in \mathcal{S}_H^{*0}(\alpha)$.*
- *[42] If $\displaystyle\sum_{n=2}^{\infty} \frac{n^2(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n^2(n+\alpha)}{1-\alpha}|b_n| \leq 1$ and $F \in \mathcal{K}_H^0$, then $f * F \in \mathcal{K}_H^0$.*
- *[42] Let $\displaystyle\sum_{n=2}^{\infty} n^3|a_n| \leq 1$ and $F \in \mathcal{K}_H$, if $f * F$ is locally univalent, then $f * F \in \mathcal{C}_H$.*

*If $f = h + \overline{g} \in \mathcal{S}_H^0$ maps $\mathbb{D}$ onto the right half-plane $\{w : \mathbf{Re}\, w > \frac{1}{2}\}$, then it must satisfy in $h + g = \dfrac{z}{1-z}$.*

The collection of functions $f = h + \overline{g}$ that map $\mathbb{D}$ onto the right half-plane $R = \{w : \mathbf{Re}\ w > -\dfrac{1}{2}\}$, have the form

$$h(z) + g(z) = \frac{z}{1-z}$$

that $\frac{z}{1-z}$ is the extremal function for class $\mathcal{K}$, and those that map $\mathbb{D}$ onto the vertical strip, $R = \{w : \dfrac{\alpha-\pi}{2\sin\alpha} < \mathbf{Re}\ w < \dfrac{\alpha}{2\sin\alpha}\}$, have the form

$$h(z) + g(z) = \frac{1}{2i\sin\alpha}\log\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}$$

**Lemma 3.9.5.** *(Dorff [21]) Let $f_1 = h_1 + \overline{g_1} \in \mathcal{K}_H^0$ with $h_1 + g_1 = \dfrac{z}{1-z}$ and $f_2 = h_2 + \overline{g_2} \in \mathcal{K}_H^0$ with $h_2 + g_2 = \dfrac{z}{1-z}$. If $f_1 * f_2$ be locally univalent and sense-preserving, then $f_1 * f_2 \in \mathcal{S}_H^0$ and is convex in the direction of the real axis.*

## 3.10 $\mathcal{T}_H$ class

Let $\mathcal{T}_H(\alpha)$ denote the subclass of $\mathcal{S}_H$ consisting of harmonic functions $f = h + \overline{g}$ whose nonzero coefficients in expansion series of $h$, from the second on, are negative. That is,

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \quad , \quad g(z) = \sum_{n=1}^{\infty}|b_n|z^n$$

and satisfy in following condotion also,

$$\frac{\partial}{\partial\theta}\arg\{f(re^{i\theta})\} \geq \alpha$$

where $|z| = r < 1$ and $0 \leq \alpha < 1$. This class introduced by Jahangiri [38] in 1999.

Ezhilarasi R.and Sudharsan [26] investigated the class $\mathcal{S}_H^*(\phi,\psi,\lambda,\gamma,k)$ of harmonic function in $\mathcal{S}_H$ class that satisfying the condition

$$\mathbf{Re}\Big(\frac{(1+ke^{i\alpha})\big(z(h*\phi)' - \overline{z(g*\psi)'}\big)}{z'\big((1-\lambda)z + \lambda((h*\phi) + \overline{(g*\psi)}\big)} - ke^{i\alpha}\Big) \geq \gamma$$

for all real $\alpha$, and $\phi(z) = z + \sum_{n=2}^{\infty}\lambda_n z^n$, $\psi(z) = z + \sum_{n=2}^{\infty}\mu_n z^n$ are analytic with condition $\lambda_n \geq 0$, $\mu_n \geq 0$, $0 \leq \lambda \leq 1$, $z' = \dfrac{\partial}{\partial\theta}(z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 \leq \gamma < 1$. Also, let $\overline{\mathcal{S}_H^*}(\phi,\psi,\lambda,\gamma,k)$ denote the subclass of $\mathcal{S}_H^*(\phi,\psi,\lambda,\gamma,k)$ consisting of functions $f = h + \overline{g} \in \mathcal{T}_H$.

**Remark 3.10.1.** *For* $\alpha = 0$ , $\overline{\mathcal{S}_H^*}(\frac{z}{1-z}, \frac{z}{1-z}, 1, \gamma, 1) = \mathcal{T}_H(\frac{1+\gamma}{2})$.

**Lemma 3.10.1.** *[26] Let the function* $f = h + g$ *be so that* $h$ *and* $g$ *are given by (3.2), and let*

$$\sum_{n=2}^{\infty}\Big(\frac{n(1+k)-\lambda(k+\gamma)}{1-\gamma}\Big)\lambda_n|a_n| + \sum_{n=1}^{\infty}\Big(\frac{n(1+k)+\lambda(k+\gamma)}{1-\gamma}\Big)\mu_n|b_n| \leq 1$$

*where* $k \geq 0$, $\lambda_n \geq 0$, $\mu_n \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \gamma \leq 1$, $\alpha$ *is real number and if*

$$n(1-\gamma) \leq \big(n(1+k)-\lambda(k+\gamma)\big)\lambda_n \leq \big(n(1+k)+\lambda(k+\gamma)\big)\mu_n$$

*Then* $f$ *is sense preserving, harmonic univalent mapping in* $\mathbb{D}$ *and for* $\lambda = \dfrac{1-\gamma}{1+\gamma}$, $f \in \mathcal{S}_H^*(\phi, \psi, \lambda, \gamma, k)$.

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